On the Separability of the $H^{1/2}$ Seminorm on Convex Polyhedral Domains

B. Kiss * A.Krebsz † K. Szalay ‡

Submitted to HEJ. Manuscript no.: ANM-010227-A

This version is for internal use only!

Abstract

In this paper the generalization of the separability property of the $H^{1/2}$ seminorm is given for the boundary of convex polyhedral domains. Using this property, the $H^{1/2}$ seminorm on the surfaces of three-dimensional bounded domains can be represented as a simple circulant sparse matrix, which contains only $O(N\log(N))$ nonzero entries, where N denotes the number of unknowns.

1 Introduction

Let $\Omega \in R^3$ a given bounded domain. The separability property of the $H^{1/2}$ seminorm on the boundary $\partial\Omega$ means that it is spectrally equivalent to the sum of the 'partial' seminorms corresponding to the directions x,y and z, that is

$$C_1 \cdot \mid f \mid^2_{H^{1/2}(\partial\Omega)} \le \sum_{p\in\{x,y,z\}} \mid f \mid^2_{H_p^{1/2}(\partial\Omega)} \le C_2 \cdot \mid f \mid^2_{H^{1/2}(\partial\Omega)}$$

where C_1 and C_2 are positive constants independent of f.

This property was proved at first in the case of hypercubes (cf. Lemma 5.3 in [14] or [5]). A different proof for general rectangular domains and its discrete counterpart in the space of bilinear finite elements have been given in [10]. The generalization of the property to triangular domains and its discrete equivalent in the space of linear finite elements have been discussed in [11].

The purpose of this paper is to give the generlization of this property to a wide class of convex polyhedral domains. The proof is based on a special covering of the convex domains and the application of the separability property on convex poligonal domains proved in [12].

The matrix representations of the $H^{1/2}$ seminorm are efficient preconditioners for elliptic problems and boundary integral equations of first kind. Numerous papers are devoted to this topic, see for example [1, 3, 4, 5, 6, 7, 8, 13, 16].

By the use of the separability property, the $H^{1/2}$ seminorm can be represented as a sum of one-dimensional seminorms in finite element spaces. Hence the $H^{1/2}$ seminorm on the surfaces of three-dimensional bounded domains can be represented as a simple sparse circulant matrix, which contains only $O(N\log(N))$ nonzero entries, where N denotes the number of unknowns. The construction of

*Department of Mathematics, Széchenyi István College, Hédervári u. 3. H-9026 Győr, Hungary E-mail: bkiss@nimrod.math.szif.hu

†Department of Numerical Analysis, Eötvös Loránd University, Múzeum krt. 6-8. H-1088 Budapest, Hungary E-mail: krebsz@cs.elte.hu

‡Department of Mathematics, Széchenyi István College, Hédervári u. 3. H-9026 Győr, Hungary E-mail: kszalay@nimrod.math.szif.hu

this matrix representation and its application as a Schur complement preconditioner in the case of brick shaped and tetrahedral domains are discussed in [10] and [11], respectively. The presented generalization of the separabiliy property of the $H^{1/2}$ seminorm allows to generalize this preconditioner construction to the boundary of a wide class of convex polyhedral domains.

2 The Separability Property

Let $\partial\Omega$ denote the boundary of a given bounded convex polyhedron shaped domain. Define the 'main' directional unit vectors

$$\begin{array}{lll} \underline{v}_1 = (1,0,0)^T\,, & \underline{v}_2 = (0,1,0)^T\,, & \underline{v}_3 = (0,0,1)^T\,, \\ \underline{v}_4 = (-1,0,0)^T\,, & \underline{v}_5 = (0,-1,0)^T\,, & \underline{v}_6 = (0,0,-1)^T\,, \end{array}$$

which are parallel to the coordinate axes and the 'supplementary' directional unit vectors

$$\begin{array}{lll} \underline{v}_7 = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)^T, & \underline{v}_8 = \left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)^T, & \underline{v}_9 = \left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^T, \\ \underline{v}_{10} = \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)^T, & \underline{v}_{11} = \left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)^T, & \underline{v}_{12} = \left(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)^T, \\ \underline{v}_{13} = \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)^T, & \underline{v}_{14} = \left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)^T, & \underline{v}_{15} = \left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)^T, \\ \underline{v}_{16} = \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)^T, & \underline{v}_{17} = \left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)^T, & \underline{v}_{18} = \left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^T. \end{array}$$

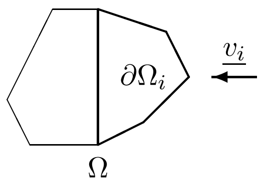

Figure 1

Assume that Ω is not transparent and let $\partial\Omega_i$ denote that set of points of $\partial\Omega$ which are visible from the direction $\underline{v}_i$. So we get the covering

$$\partial\Omega = \cup_{i=1}^{18} \partial\Omega_i \tag{1}$$

of $\partial\Omega$.

For the sake of simplicity of the notations, some index sets are introduced. These sets are given in chart form and reviewed now.

The index set I contains the indices of the unit vectors $\underline{v}_i$ defined above.

$I =$ (2)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

We introduce local coordinate systems on the sets $\partial\Omega_i$ $(i \in I)$ determined by the basis vectors $\underline{v}_i, \underline{v}_j, \underline{v}_k$. I_{loc} denotes the set of the indices of these basis vectors.

$I_{loc} =$ (3)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
j	2	1	1	2	1	1	3	2	1	3	2	1	3	2	1	3	2	1
k	3	3	2	3	3	2	13	14	15	13	14	15	7	8	9	7	8	9

The elements of I_x are the indices of these vectors $\underline{v}_i$ which are perpendicular to the axis x (See Figure 2.). $I_{loc,x}$ contains the indices of the basis vectors belonging to the local coordinate systems introduced on the sets $\partial\Omega_i$ $(i \in I_x)$, where the components j and k are ordered so that $\underline{v}_j$ is the direction of the axis x. The sets I_y, $I_{loc,y}$ and I_z, $I_{loc,z}$ are defined analogously.

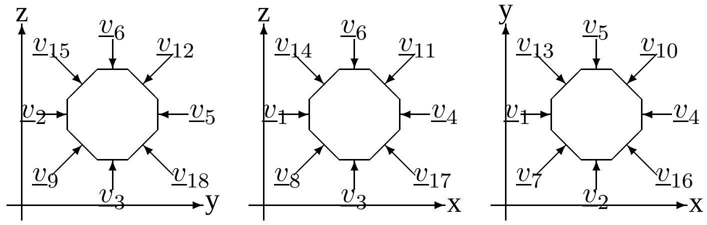

Figure 2.

$$I_x = \begin{array}{|c||c|c|c|c|c|c|c|c|}\hline i & 2 & 9 & 3 & 18 & 5 & 12 & 6 & 15 \\ \hline\end{array}, \quad I_{loc,x} = \begin{array}{|c||c|c|c|c|c|c|c|c|}\hline i & 2 & 9 & 3 & 18 & 5 & 12 & 6 & 15 \\ \hline j & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline k & 3 & 15 & 2 & 9 & 3 & 15 & 2 & 9 \\ \hline\end{array} \tag{4}$$

$$I_y = \begin{array}{|c||c|c|c|c|c|c|c|c|}\hline i & 1 & 8 & 3 & 17 & 4 & 11 & 6 & 14 \\ \hline\end{array}, \quad I_{loc,y} = \begin{array}{|c||c|c|c|c|c|c|c|c|}\hline i & 1 & 8 & 3 & 17 & 4 & 11 & 6 & 14 \\ \hline j & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \hline k & 3 & 14 & 1 & 8 & 3 & 14 & 1 & 8 \\ \hline\end{array} \tag{5}$$

$$I_z = \begin{array}{|c||c|c|c|c|c|c|c|c|}\hline i & 1 & 7 & 2 & 16 & 4 & 10 & 5 & 13 \\ \hline\end{array}, \quad I_{loc,z} = \begin{array}{|c||c|c|c|c|c|c|c|c|}\hline i & 1 & 7 & 2 & 16 & 4 & 10 & 5 & 13 \\ \hline j & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline k & 2 & 13 & 1 & 7 & 2 & 13 & 1 & 7 \\ \hline\end{array} \tag{6}$$

The sets $\partial\Omega_i$ $(i \in I_p)$ give a covering of $\partial\Omega$ around the axis p $(p \in \{x, y, z\})$. The index sets $I_{nebo,x}$, $I_{nebo,y}$ and $I_{nebo,z}$ are used to make the 'redundant' partial seminorms 'disappear'. These index sets contain the indices of the 'supplementary' directions and their left and right neighbours belonging to the covering of $\partial\Omega$ around the coordinate axes. The elements of $I_{nebo,p}$ $(p \in \{x, y, z\})$, that is the indices of the left $\underline{v}_j$ and the right $\underline{v}_k$ neighbour of the 'supplementary' direction $\underline{v}_i$ can be read easily from Figure 2.

$$I_{nebo,x} = \begin{array}{|c||c|c|c|c|}\hline i & 9 & 18 & 12 & 15 \\ \hline l & 2 & 3 & 5 & 6 \\ \hline r & 3 & 5 & 6 & 2 \\ \hline\end{array}, \quad I_{nebo,y} = \begin{array}{|c||c|c|c|c|}\hline i & 8 & 17 & 11 & 14 \\ \hline l & 1 & 3 & 4 & 6 \\ \hline r & 3 & 4 & 6 & 1 \\ \hline\end{array}, \quad I_{nebo,z} = \begin{array}{|c||c|c|c|c|}\hline i & 7 & 16 & 10 & 13 \\ \hline l & 1 & 2 & 4 & 5 \\ \hline r & 2 & 4 & 5 & 1 \\ \hline\end{array} \tag{7}$$

The proof of the separability theorem needs projections. We introduce and investigate these projections in next.

Let us write the points of $\partial\Omega_i$ $(i \in I)$ in the local coordinate system $\underline{v}_i, \underline{v}_j, \underline{v}_k$ determined by the indices $(i, j, k) \in I_{loc}$ into the form

$$\underline{x} = \alpha\underline{v}_i + \beta\underline{v}_j + \gamma\underline{v}_k \quad \forall\underline{x} \in \partial\Omega_i \tag{8}$$

and define the projections $P_i : \partial\Omega_i \to R^2$,

$$P_i(\underline{x}) = (\beta, \gamma)^T, \quad \forall\underline{x} \in \partial\Omega_i. \tag{9}$$

Introduce the image sets

$$P_i(\partial\Omega_i) = \left\{P_i(\underline{x}) \in R^2 \mid \underline{x} \in \partial\Omega_i\right\}, \quad (i \in I), \tag{10}$$

$$P_i(\partial\Omega_i \cap \partial\Omega_l) = \left\{P_i(\underline{x}) \in R^2 \mid \underline{x} \in \partial\Omega_i \cap \partial\Omega_l\right\}, \tag{11}$$

and

$$P_i(\partial\Omega_i \cap \partial\Omega_r) = \left\{P_i(\underline{x}) \in R^2 \mid \underline{x} \in \partial\Omega_i \cap \partial\Omega_r\right\}, \tag{12}$$

$(i, l, r) \in I_{nebo,x} \cup I_{nebo,y} \cup I_{nebo,x}$ and their normal domain forms

$$P_i(\partial\Omega_q) = \left\{(\beta, \gamma)^T \in R^2 \mid \check{\beta}_q \le \beta \le \hat{\beta}_q \text{ and } \check{\varphi}_q(\beta) \le \gamma \le \hat{\varphi}_q(\beta)\right\} = \tag{13}$$

$$= \left\{ (\beta,\gamma)^T \in R^2 \mid \check{\gamma}_q \le \gamma \le \hat{\gamma}_q \text{ and } \check{\psi}_q(\gamma) \le \beta \le \hat{\psi}_q(\gamma) \right\},$$

where we used the notation

$$\partial\Omega_q = \begin{cases} \partial\Omega_i & \text{if } q = i \\ \partial\Omega_i \cap \Omega_l & \text{if } q = i,l \\ \partial\Omega_i \cap \Omega_r & \text{if } q = i,r \end{cases}.$$

The proof of the three-dimensional separability property is based on the application of its two-dimensional counterpart [12]. For the application of this two-dimensional counterpart we must assume that the sets $P_i(\partial\Omega_q)$ satisfy the condition

$$A1: \quad \check{\varphi}_q(\beta_1) \le \hat{\varphi}_q(\beta_2), \quad \forall \beta_1,\beta_2 \in [\check{\beta}_q, \hat{\beta}_q], \tag{14}$$

or

$$A2: \quad \check{\psi}_q(\gamma_1) \le \hat{\psi}_q(\gamma_2), \quad \forall \gamma_1,\gamma_2 \in [\check{\gamma}_q, \hat{\gamma}_q]. \tag{15}$$

The properties of the sets Ω_i and the projections P_i used subsequently are summarized into the following lemma.

Lemma 2.1 *The sets $\partial\Omega_i$ $(i \in I)$ can be written into the form*

$$\partial\Omega_i = \left\{ \alpha(\beta,\gamma)\underline{v}_i + \beta\underline{v}_j + \gamma\underline{v}_k \quad | \quad (\beta,\gamma) \in P_i(\partial\Omega_i) \right\} \tag{16}$$

and hence the projections P_i are invertible.

The coordinate α as a function of the coordinates β and γ is piecewise continuously differentiable, and there exists positive constant C_1 such that

$$||| \nabla\alpha(\beta,\gamma) ||| \le C_1 \tag{17}$$

for almost every $(\beta,\gamma) \in P_i(\partial\Omega_i)$, where

$$||| \nabla\alpha(\beta,\gamma) ||| = \max_{|b|^2+|c|^2=1,\ b,c\in R} \left| \frac{\partial\alpha}{\partial\beta}(\beta,\gamma)\cdot b + \frac{\partial\alpha}{\partial\gamma}(\beta,\gamma)\cdot c \right|$$

Proof: The first statement of the lemma is a straightforward consequence of the definitions of the sets $\partial\Omega_i$ and the projections P_i.

Since Ω is a polyhedron-shaped domain, the function $\alpha(\beta,\gamma)$ is piecewise linear. Hence $\alpha(\beta,\gamma)$ is piecewise continuously differentiable and the second statement of the lemma holds. ■

The formulation of the three-dimensional separability property needs the introduction of the following seminorms on the sets $\partial\tilde{\Omega} \in \{\partial\Omega, \partial\Omega_q\}$:

The $H^{1/2}$ seminorm is defined in the usual way by the formula

$$| f |^2_{H^{1/2}(\partial\tilde{\Omega})} = \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \frac{| f(\underline{x}) - f(\underline{y}) |^2}{\|\underline{x}-\underline{y}\|^3} ds(\underline{y})ds(\underline{x}), \tag{18}$$

where $s(\cdot)$ is the area on $\partial\tilde{\Omega}$ and

$$\|\underline{x}-\underline{y}\| = \left((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 \right)^{1/2}$$

denotes the three-dimensional euclidean distance.

The 'partial' seminorm belonging to the direction x is defined by the expression

$$| f |^2_{H_x^{1/2}(\partial\tilde{\Omega})} = \tag{19}$$

$$\int_{x_{min}}^{x_{max}} \int_{\partial\Omega_x} \int_{\partial\tilde{\Omega}_x} \frac{\mid f(x,y_1,z_1) - f(x,y_2,z_2) \mid^2}{\mid (y_1-y_2)^2 + (z_1-z_2)^2 \mid} ds_x(y_2,z_2) ds_x(y_1,z_1) dx,$$

where

$$x_{min} = \min_{(\tilde{x},\tilde{y},\tilde{z})^T \in \partial\tilde{\Omega}} \tilde{x}, \quad x_{max} = \max_{(\tilde{x},\tilde{y},\tilde{z})^T \in \partial\tilde{\Omega}} \tilde{x},$$

$$\partial\tilde{\Omega}_x = \left\{ (\tilde{x},\tilde{y},\tilde{z})^T \in \partial\tilde{\Omega} \quad \mid \quad \tilde{x} = x \right\}$$

and $s_x(.)$ is the arclength on $\partial\tilde{\Omega}_x$.

The seminorms belonging to the directions y and z are defined analogously by the formulas

$$\mid f \mid^2_{H_y^{1/2}(\partial\tilde{\Omega})} = \tag{20}$$

$$\int_{y_{min}}^{y_{max}} \int_{\partial\tilde{\Omega}_y} \int_{\partial\tilde{\Omega}_y} \frac{\mid f(x_1,y,z_1) - f(x_2,y,z_2) \mid^2}{\mid (x_1-x_2)^2 + (z_1-z_2)^2 \mid} ds_y(x_2,z_2) ds_y(x_1,z_1) dy,$$

and

$$\mid f \mid^2_{H_z^{1/2}(\partial\tilde{\Omega})} = \tag{21}$$

$$\int_{z_{min}}^{z_{max}} \int_{\partial\tilde{\Omega}_z} \int_{\partial\tilde{\Omega}_z} \frac{\mid f(x_1,y_1,z) - f(x_2,y_2,z) \mid^2}{\mid (x_1-x_2)^2 + (y_1-y_2)^2 \mid} ds_z(x_2,y_2) ds_z(x_1,y_1) dz.$$

The three-dimensional separability property can be formulated as follows:

Theorem 2.2 *Let Ω be a bounded convex polyhedron-shaped domain. Assume that each $P_i(\partial\Omega_q)$ defined previously satisfies the condition $A1$ or $A2$ and there exist positive constants C_{21} and C_{22} independent of f, such that*

$$C_{21} \cdot \mid f \mid^2_{H^{1/2}(\partial\Omega)} \le \sum_{i \in I} \mid f \mid^2_{H^{1/2}(\partial\Omega_i)} \le C_{22} \cdot \mid f \mid^2_{H^{1/2}(\partial\Omega)} \tag{22}$$

for all $f \in H^{1/2}(\partial\Omega)$,

$$C_{21} \cdot \mid f \mid^2_{H_p^{1/2}(\partial\Omega)} \le \sum_{i \in I_p} \mid f \mid^2_{H_p^{1/2}(\partial\Omega_i)} \le C_{22} \cdot \mid f \mid^2_{H_p^{1/2}(\partial\Omega)} \tag{23}$$

for all $f \in H_p^{1/2}(\partial\Omega)$ and $p \in \{x,y,z\}$.

Then there exist positive constants C_{23} and C_{24} independent of f such that

$$C_{23} \cdot \mid f \mid^2_{H^{1/2}(\partial\Omega)} \le \sum_{p \in \{x,y,z\}} \mid f \mid^2_{H_p^{1/2}(\partial\Omega)} \le C_{24} \cdot \mid f \mid^2_{H^{1/2}(\partial\Omega)} \tag{24}$$

for all $f \in H^{1/2}(\partial\Omega)$.

Due to the condition of the theorem concerning the covering of $\partial\Omega$ the proof of the equivalence of the seminorm $H^{1/2}(\partial\Omega)$ and the 'partial' seminorms $H_p^{1/2}(\partial\Omega)$ $(p \in \{x,y,z\})$ can be reduced to the proof of the equivalence of the $H^{1/2}(\partial\Omega_i)$ seminorm and the 'partial' seminorms $H_p^{1/2}(\partial\Omega_i)$ $(p \in \{x,y,z\})$.

Define the seminorms

$$\mid f \mid^2_{H^{1/2}(P_i(\partial\Omega_q))} = \tag{25}$$

$$\int_{P_i(\partial\Omega_q)} \int_{P_i(\partial\Omega_q)} \frac{\mid f(P_i^{-1}(\beta_1,\gamma_1)) - f(P_i^{-1}(\beta_2,\gamma_2)) \mid^2}{\mid (\beta_1-\beta_2)^2 + (\gamma_1-\gamma_2)^2 \mid^{3/2}} d\gamma_2 d\beta_2 d\gamma_1 d\beta_1,$$

and

$$| f |^2_{H^{1/2}_{jk}(P_i(\partial\Omega_q))} = \tag{26}$$

$$\int_{\check{\beta}_q}^{\hat{\beta}_q} \int_{\check{\varphi}_q(\beta)}^{\hat{\varphi}_q(\beta)} \int_{\check{\varphi}_q(\beta)}^{\hat{\varphi}_q(\beta)} \frac{| f(P_i^{-1}(\beta,\gamma_1)) - f(P_i^{-1}(\beta,\gamma_2)) |^2}{| \gamma_1 - \gamma_2 |^2} d\gamma_2 d\gamma_1 d\beta,$$

$$| f |^2_{H^{1/2}_{kj}(P_i(\partial\Omega_q))} = \tag{27}$$

$$\int_{\check{\gamma}_q}^{\hat{\gamma}_q} \int_{\check{\psi}_q(\gamma)}^{\hat{\psi}_q(\gamma)} \int_{\check{\psi}_q(\gamma)}^{\hat{\psi}_q(\gamma)} \frac{| f(P_i^{-1}(\beta_1,\gamma)) - f(P_i^{-1}(\beta_2,\gamma)) |^2}{| \beta_1 - \beta_2 |^2} d\beta_2 d\beta_1 d\gamma$$

on the sets $P_i(\partial\Omega_q)$.

By the use of the next lemma the proof of the equivalence of the $H^{1/2}(\partial\Omega_i)$ seminorm and the 'partial' seminorms $H^{1/2}_p(\partial\Omega_i)$ $(p \in \{x, y, z\})$ can be simplified to the proof of the equivalence of the just intruduced $H^{1/2}(P_i(\partial\Omega_i))$ seminorm and 'partial' seminorms $H^{1/2}_{jk}(P_i(\partial\Omega_i))$ and $H^{1/2}_{kj}(P_i(\partial\Omega_i))$.

Lemma 2.3 *Under the hypotheses of the theorem there exist positive constants C_{31} and C_{32} independent of f, such that*

$$C_{31} \cdot | f |^2_{H^{1/2}(\partial\Omega_i)} \quad \le \quad | f |^2_{H^{1/2}(P_i(\partial\Omega_i))} \quad \le \quad C_{32} \cdot | f |^2_{H^{1/2}(\partial\Omega_i)} \tag{28}$$

for all $i \in I$,

$$C_{31} \cdot | f |^2_{H^{1/2}(\partial\Omega_q)} \quad \le \quad | f |^2_{H^{1/2}(P_i(\partial\Omega_q))} \quad \le \quad C_{32} \cdot | f |^2_{H^{1/2}(\partial\Omega q)} \tag{29}$$

for all $(i, l, r) \in I_{nebo,x} \cup I_{nebo,y} \cup I_{nebo,z}$, where $q \in \{(i, l), (i, r)\}$, and

$$C_{31} \cdot | f |^2_{H^{1/2}_p(\partial\Omega_i)} \quad \le \quad | f |^2_{H^{1/2}_{jk}(P_i(\partial\Omega_i))} \quad \le \quad C_{32} \cdot | f |^2_{H^{1/2}_p(\partial\Omega_i)} \tag{30}$$

for all $i \in I_p$ és $(i, j, k) \in I_{loc,p}$ and $p \in \{x, y, z\}$.

Proof: We prove only the first equivalence, the others can be verified analogously.

Let us consider the equivalent form

$$| f |^2_{H^{1/2}(\partial\Omega_i)} =$$

$$\int_{P_i(\partial\Omega_i)} \int_{P_i(\partial\Omega_i)} \frac{| f(P_i^{-1}(\beta_1,\gamma_1)) - f(P_i^{-1}(\beta_2,\gamma_2)) |^2}{\| P_i^{-1}(\beta_1,\gamma_1) - P_i^{-1}(\beta_2,\gamma_2) \|^3} ds(P_i^{-1}(\beta_2,\gamma_2)) ds(P_i^{-1}(\beta_1,\gamma_1))$$

of $| f |^2_{H^{1/2}(\partial\Omega_i)}$, where

$$\| P_i^{-1}(\beta_1,\gamma_1) - P_i^{-1}(\beta_2,\gamma_2) \|^2 = (\alpha(\beta_1,\gamma_1) - \alpha(\beta_2,\gamma_2))^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_1)^2$$

and

$$ds(P_i^{-1}(\beta_l,\gamma_l)) = \left(1 + \left(\frac{\partial\alpha}{\partial\beta}\right)^2 (\beta_l,\gamma_l) + \left(\frac{\partial\alpha}{\partial\gamma}\right)^2 (\beta_l,\gamma_l)\right)^{1/2} d\gamma_l d\beta_l, \quad (l = 1, 2).$$

According to Lemma 2.1 there holds $||| \nabla\alpha(\beta_l,\gamma_l) ||| \le C_1$ for almost every $(\beta_l,\gamma_l) \in P_i(\Omega_i)$, and thus the previous two terms can be estimated as

$$\left((\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_1)^2\right) \le$$

$$\le \| P_i^{-1}(\beta_1,\gamma_1) - P_i^{-1}(\beta_2,\gamma_2) \|^2 \le$$

$$\leq (1+C_1^2)((\beta_1-\beta_2)^2+(\gamma_1-\gamma_1)^2)$$

and

$$1 d\gamma_l d\beta_l \leq ds(P_i^{-1}(\beta_l,\gamma_l)) \leq (1+2\cdot C_1^2)^{1/2} d\gamma_l d\beta_l.$$

So the statement of the lemma holds with the choice $C_{31}=\frac{1}{(1+2\cdot C_1^2)}$ and $C_{32}=(1+C_1^2)^{3/2}$. ■

Since the two-dimensional sets $P_i(\partial\Omega_q)$ satisfy the condition $A1$ or $A2$, we can apply the two-dimensional separability theorem [12], and we get the following equivalence between the seminorms defined on the sets $P_i(\partial\Omega_q)$:

Lemma 2.4 *Under the hypotheses of the theorem there exist positive constants C_{41} and C_{42} independent of f such that*

$$C_{41}\cdot \mid f\mid^2_{H^{1/2}(P_i(\partial\Omega_i))} \leq \mid f\mid^2_{H^{1/2}_{jk}(P_i(\partial\Omega_i))} + \mid f\mid^2_{H^{1/2}_{kj}(P_i(\partial\Omega_i))} \leq C_{42}\cdot \mid f\mid^2_{H^{1/2}(P_i(\partial\Omega_i))} \tag{31}$$

for all $i\in I$ és $(i,j,k)\in I_{loc}$, moreover

$$C_{41}\cdot \mid f\mid^2_{H^{1/2}(P_i(\partial\Omega_q))} \leq \mid f\mid^2_{H^{1/2}_{jk}(P_i(\partial\Omega_q))} + \mid f\mid^2_{H^{1/2}_{kj}(P_i(\partial\Omega_q))} \leq C_{42}\cdot \mid f\mid^2_{H^{1/2}(P_i(\partial\Omega_q))} \tag{32}$$

for all $(i,l,r)\in I_{nebo,x}\cup I_{nebo,y}\cup I_{nebo,z}$ and $(i,j,k)\in I_{loc}$, where $q\in\{(i,l),(i,r)\}$.

The next lemma serves for the 'disappearing' of the 'redundant' partial seminorms.

Lemma 2.5 *Under the hypotheses of the theorem there exists a positive constant C_5 independent of f such that*

$$\mid f\mid^2_{H^{1/2}_{kj}(P_i(\partial\Omega_i))} \leq \tag{33}$$

$$\leq C_5\cdot\left(\mid f\mid^2_{H^{1/2}_{mn}(P_l(\partial\Omega_l))} + \mid f\mid^2_{H^{1/2}_{nm}(P_l(\partial\Omega_l))} + \mid f\mid^2_{H^{1/2}_{st}(P_r(\partial\Omega_r))} + \mid f\mid^2_{H^{1/2}_{ts}(P_r(\partial\Omega_r))}\right)$$

for all $(i,l,r)\in I_{nebo,p}$, $(i,j,k),(l,m,n),(r,s,t)\in I_{loc,p}$ and $p\in\{x,y,z\}$.

Proof: The definition of the sets $\partial\Omega_q$ implies that

$$\partial\Omega_i \subseteq (\partial\Omega_i\cap\partial\Omega_l)\cup(\partial\Omega_i\cap\partial\Omega_r).$$

Hence

$$P_i(\partial\Omega_i)\subseteq P_i(\partial\Omega_i\cap\partial\Omega_l)\cup P_i(\partial\Omega_i\cap\partial\Omega_r)$$

and so

$$\mid f\mid^2_{H^{1/2}_{kj}(P_i(\partial\Omega_i))} \leq \mid f\mid^2_{H^{1/2}_{kj}(P_i(\partial\Omega_i\cap\partial\Omega_l))} + \mid f\mid^2_{H^{1/2}_{kj}(P_i(\partial\Omega_i\cap\partial\Omega_r))} = I_1+I_2.$$

Apply Lemma 2.3 and Lemma 2.4 to the therm I_1 to obtain

$$I_1\leq C_{42}\cdot \mid f\mid^2_{H^{1/2}(P_i(\partial\Omega_i\cap\partial\Omega_l))} \leq C_{32}\cdot C_{42}\cdot \mid f\mid^2_{H^{1/2}(\partial\Omega_i\cap\partial\Omega_l)} \leq$$

$$\leq C_{32}\cdot C_{42}\cdot \mid f\mid^2_{H^{1/2}(\partial\Omega_l)} \leq \frac{C_{32}\cdot C_{42}}{C_{31}}\cdot \mid f\mid^2_{H^{1/2}(P_l(\partial\Omega_l))} \leq$$

$$\leq \frac{C_{32}\cdot C_{42}}{C_{31}\cdot C_{41}}\cdot\left(\mid f\mid^2_{H^{1/2}_{mn}(P_l(\partial\Omega_l))} + \mid f\mid^2_{H^{1/2}_{nm}(P_l(\partial\Omega_l))}\right)$$

The therm I_2 can be estimated analogously:

$$I_2\leq \frac{C_{32}\cdot C_{42}}{C_{31}\cdot C_{41}}\cdot\left(\mid f\mid^2_{H^{1/2}_{st}(P_r(\partial\Omega_r))} + \mid f\mid^2_{H^{1/2}_{ts}(P_r(\partial\Omega_r))}\right)$$

Hence the lemma holds with the choice $C_5 = \frac{C_{32} \cdot C_{42}}{C_{31} \cdot C_{41}}$. ■

Proof of Theorem 2.2: During the proof of the theorem, D denote positive constants independent of f.

Due to the covering condition of the theorem it is enough to prove the equivalence

$$D_{11} \cdot \sum_{i \in I} \mid f \mid^2_{H^{1/2}(\partial\Omega_i)} \leq \sum_{p \in \{x,y,z\}} \left(\sum_{i \in I_p} \mid f \mid^2_{H_p^{1/2}(\partial\Omega_i)} \right) \leq D_{12} \cdot \sum_{i \in I} \mid f \mid^2_{H^{1/2}(\partial\Omega_i)} .$$

Applying Lemma 2.3 this can be reduced to proof of the equivalence

$$D_{21} \cdot \sum_{i \in I} \mid f \mid^2_{H^{1/2}(P_i(\partial\Omega_i))} \leq$$

$$\leq \sum_{p \in \{x,y,z\}} \left(\sum_{(i,j,k) \in I_{loc,p}} \mid f \mid^2_{H_{jk}^{1/2}(P_i(\partial\Omega_i))} \right) \leq$$

$$\leq D_{22} \cdot \sum_{i \in I} \mid f \mid^2_{H^{1/2}(P_i(\partial\Omega_i))} .$$

Using Lemma 2.3 our proof can be simplified to verify the equivalence

$$D_{31} \cdot \sum_{(i,j,k) \in I_{loc}} \left(\mid f \mid^2_{H_{jk}^{1/2}(P_i(\partial\Omega_i))} + \mid f \mid^2_{H_{kj}^{1/2}(P_i(\partial\Omega_i))} \right) \leq$$

$$\leq \sum_{p \in \{x,y,z\}} \left(\sum_{(i,j,k) \in I_{loc,p}} \mid f \mid^2_{H_{jk}^{1/2}(P_i(\partial\Omega_i))} \right) \leq$$

$$\leq D_{32} \cdot \sum_{(i,j,k) \in I_{loc}} \left(\mid f \mid^2_{H_{jk}^{1/2}(P_i(\partial\Omega_i))} + \mid f \mid^2_{H_{kj}^{1/2}(P_i(\partial\Omega_i))} \right) .$$

Taking into account the identity

$$\sum_{(i,j,k) \in I_{loc}} \left(\mid f \mid^2_{H_{jk}^{1/2}(P_i(\partial\Omega_i))} + \mid f \mid^2_{H_{kj}^{1/2}(P_i(\partial\Omega_i))} \right) =$$

$$= \sum_{p \in \{x,y,z\}} \left(\sum_{(i,j,k) \in I_{loc,p}} \mid f \mid^2_{H_{jk}^{1/2}(P_i(\partial\Omega_i))} \right) +$$

$$+ \sum_{p \in \{x,y,z\}} \left(\sum_{(i,l,r) \in I_{nebo,p}, (i,j,k) \in I_{loc,p}} \mid f \mid^2_{H_{kj}^{1/2}(P_i(\partial\Omega_i))} \right)$$

and the definitions of $I_{loc,p}$ and $I_{nebo,p}$, the previous equivalence follows from the estimation

$$\sum_{p \in \{x,y,z\}} \left(\sum_{(i,l,r) \in I_{nebo,p}, (i,j,k) \in I_{loc,p}} \mid f \mid^2_{H_{kj}^{1/2}(P_i(\partial\Omega_i))} \right) \leq$$

$$\leq C_5 \cdot \sum_{p \in \{x,y,z\}} \left(\sum_{(i,l,r) \in I_{nebo,p}, (l,m,n) \in I_{loc,p}} \left(\mid f \mid^2_{H_{mn}^{1/2}(P_l(\partial\Omega_l))} + \mid f \mid^2_{H_{nm}^{1/2}(P_l(\partial\Omega_l))} \right) \right) +$$

$$+ C_5 \cdot \sum_{p \in \{x,y,z\}} \left(\sum_{(i,l,r) \in I_{nebo,p}, (r,s,t) \in I_{loc,p}} \left(\mid f \mid^2_{H_{st}^{1/2}(P_r(\partial\Omega_r))} + \mid f \mid^2_{H_{ts}^{1/2}(P_r(\partial\Omega_r))} \right) \right) \leq$$

$$\leq 2 \cdot C_5 \cdot \sum_{p \in \{x,y,z\}} \left(\sum_{(i,j,k) \in I_{loc,p}} |\, f \,|^2_{H^{1/2}_{jk}(P_i(\partial\Omega_i))} \right),$$

where we have applied Lemma 2.5.

The proof is complete. ■

Remark 2.6 *The brick shaped domains satisfy the conditions of Theorem 2.2. For example in the case of the unit cube the sets $\partial\Omega_i$ $(i \in I)$ are the following:*

$$\partial\Omega_i = Q_i, \quad (i = 1, \dots, 6),$$

$$\begin{array}{lll} \partial\Omega_7 = Q_1 \cup Q_2, & \partial\Omega_8 = Q_1 \cup Q_3, & \partial\Omega_9 = Q_2 \cup Q_3, \\ \partial\Omega_{10} = Q_4 \cup Q_5, & \partial\Omega_{11} = Q_4 \cup Q_6, & \partial\Omega_{12} = Q_5 \cup Q_6, \\ \partial\Omega_{13} = Q_1 \cup Q_5, & \partial\Omega_{14} = Q_1 \cup Q_6, & \partial\Omega_{15} = Q_2 \cup Q_6, \\ \partial\Omega_{16} = Q_4 \cup Q_2, & \partial\Omega_{17} = Q_4 \cup Q_3, & \partial\Omega_{18} = Q_5 \cup Q_2, \end{array}$$

where Q_i denote the faces

$$\begin{array}{lll} Q_1 = \{0\} \times [0,1] \times [0,1], & Q_2 = [0,1] \times \{0\} \times [0,1], & Q_3 = [0,1] \times [0,1] \times \{0\}, \\ Q_4 = \{1\} \times [0,1] \times [0,1], & Q_5 = [0,1] \times \{1\} \times [0,1], & Q_6 = [0,1] \times [0,1] \times \{1\} \end{array}$$

of the unit cube.

Theorem 2.2. can be applied only to a quite narrow class of polyhedrons. However the following generalization allows to prove the separability property for a much wider class.

Theorem 2.7 *Let Ω be a bounded convex polyhedron-shaped domain. Assume that to the 'main' directions $\underline{v}_i$ $(i = 1, \dots, 6)$ there can be given such 'supplementary' directions*

$$\underline{v}_i = \lambda_{il} \cdot \underline{v}_l + \lambda_{ir} \cdot \underline{v}_r, \quad \forall (i,l,r) \in I_{nebo,x} \cup I_{nebo,y} \cup I_{nebo,z}, \tag{34}$$

where $\lambda_{i,l}, \lambda_{i,r} > 0$, $\lambda^2_{i,l} + \lambda^2_{i,r} = 1$, that the suitable chosen sets $\partial\tilde{\Omega}_i \subseteq \partial\Omega_i$ $(i \in I)$ satisfy the conditions

$$\partial\tilde{\Omega}_i \subseteq \partial\tilde{\Omega}_l \cup \partial\tilde{\Omega}_r, \quad \forall (i,l,r) \in I_{nebo,x} \cup I_{nebo,y} \cup I_{nebo,z}, \tag{35}$$

and for the sets $P_i(\partial\tilde{\Omega}_q)$ the condition $A1$ or $A2$ hold. Moreover assume that there exist positive constants C_{71} and C_{72} independent of f such that

$$C_{71} \cdot |\, f \,|^2_{H^{1/2}(\partial\Omega)} \leq \sum_{i \in I} |\, f \,|^2_{H^{1/2}(\partial\Omega_i)} \leq C_{72} \cdot |\, f \,|^2_{H^{1/2}(\partial\Omega)} \tag{36}$$

for all $f \in H^{1/2}(\partial\Omega)$ and

$$C_{71} \cdot |\, f \,|^2_{H^{1/2}_p(\partial\Omega)} \leq \sum_{i \in I_p} |\, f \,|^2_{H^{1/2}_p(\partial\Omega_i)} \leq C_{72} \cdot |\, f \,|^2_{H^{1/2}_p(\partial\Omega)} \tag{37}$$

for all $f \in H^{1/2}_p(\partial\Omega)$ and $p \in \{x,y,z\}$.

Then there exist positive constants C_{73} and C_{74} independent of f, such that

$$C_{73} \cdot |\, f \,|^2_{H^{1/2}(\partial\Omega)} \leq \sum_{i \in I_p} |\, f \,|^2_{H^{1/2}_p(\partial\Omega_i)} \leq C_{74} \cdot |\, f \,|^2_{H^{1/2}(\partial\Omega)} \tag{38}$$

for all $f \in H^{1/2}(\partial\Omega)$.

The theorem can be proved analogously to Theorem 2.2.

Remark 2.8 *In the case of the tetrahedron*

$$T = \left\{(x,y,z) \in R^3 \mid 0 \le x,y,z \text{ and } x+y+z \le 1\right\}$$

when the 'supplementary' directions are chosen as

$$\begin{array}{lll}
\underline{v}_7 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T, & \underline{v}_8 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T, & \underline{v}_9 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T, \\
\underline{v}_{10} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)^T, & \underline{v}_{11} = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T, & \underline{v}_{12} = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T, \\
\underline{v}_{13} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right)^T, & \underline{v}_{14} = \left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right)^T, & \underline{v}_{15} = \left(0, \frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^T, \\
\underline{v}_{16} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)^T, & \underline{v}_{17} = \left(-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)^T, & \underline{v}_{18} = \left(0, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T
\end{array}$$

the sets $\partial\Omega_i$ $(i \in I)$ are the following:

$$\partial\Omega_i = T_i, \quad (i = 1,2,3), \quad \partial\Omega_i = T_4, \quad (i = 4,5,6,10,11,12),$$

$$\partial\Omega_7 = T_1 \cup T_2, \quad \partial\Omega_8 = T_1 \cup T_3, \quad \partial\Omega_9 = T_2 \cup T_3,$$

$$\partial\Omega_{13} = T_1 \cup T_4, \quad \partial\Omega_{14} = T_1 \cup T_4, \quad \partial\Omega_{15} = T_2 \cup T_4,$$

$$\partial\Omega_{16} = T_2 \cup T_4, \quad \partial\Omega_{17} = T_3 \cup T_4, \quad \partial\Omega_{18} = T_3 \cup T_4,$$

where T_i denote the faces

$$T_1 = \{(0,y,z) \mid 0 \le y \le 1 \text{ and } 0 \le z \le 1-y\},$$

$$T_2 = \{(x,0,z) \mid 0 \le x \le 1 \text{ and } 0 \le z \le 1-x\},$$

$$T_3 = \{(x,y,0) \mid 0 \le x \le 1 \text{ and } 0 \le y \le 1-x\},$$

$$T_4 = \{(x,y,z) \mid 0 \le x,y,z \text{ and } x+y+z = 1\}$$

of T.

It can be verified by straightforward computation that the choice of the sets $\partial\tilde{\Omega}_i = \partial\Omega_i$ satisfies the conditions of Theorem 2.7 and thus the $H^{1/2}$ seminorm is separable on T.

Remark 2.9 *It is easy to check that in the case of 'egg' shaped polyhedrons and its halfs and quarters the conditions of the previous theorem holds. The conditions of the theorems holds in the case of the convex polygon based prisms. We note that in domain decomposition methods it is easy to decompose the bounded domains into such type of subdomains.*

References

[1] J. H. Bramble, J. E. Pasciak, A. H. Schatz, *The Construction of Preconditioners for Elliptic Problems by Substructuring I-IV.* Math. of Comp. Vol. 47, pp. 103–134, Vol. 49, pp. 1–16, Vol. 51, pp. 415–430, Vol. 53, pp. 1–24, (1986-1989).

[2] T.F. Chan, C.C.J Kuo, C.H. Tong, *A Domain Decomposition Preconditioner Based on a Change to a Multilevel Nodal Basis,* SIAM J. Sci. Stat. Comput., Vol. 12, No. 6, pp. 1486–1495 (1991).

[3] T.F. Chan, T. P. Mathew, *Domain Decomposition Algorithms,* Acta Numerica, pp. 61–143 (1994).

[4] R.H. Chan, M. K. Ng, *Conjugate Gradient Methods for Toeplitz Systems,* SIAM Review, Vol. 38, No. 3, pp. 427–482 (1996).

[5] M. Dryja, *A Method of Domain Decomposition for 3-D Finite Element Problems,* In: R. Glowinsky at al. (eds.): First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Society for Industrial and Applied Mathematics, Philadelphia, PA, (1988).

[6] M. Dryja, F. Smith, O. Widlund, *Schwarz Analyses for Elliptic Problems in Three Dimensions,* SIAM J. Numer. Anal., Vol. 31(6), pp. 1661-1693 (1994).

[7] B. Faermann, *Localization of the Aronszajn-Slobodeckij Norm and Application to Adaptive Boundary Element Methods,* Preprint 97-10, University of Kiel, (1997).

[8] G. Haase, B. Heise, M. Kuhn, U. Langer, *Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations*, Technical Report No 95-2, Dept. of Math., Johannes Kepler University, (1995).

[9] B. Kiss, A. Krebsz, *On the H1/2 Norm Based Schur Complement Preconditioners,* In: B.H.V. Topping (ed.): Advances in Computational Structures Technology, Civil-Comp Press, Edinburg, pp. 343–348 (1996).

[10] B. Kiss, A. Krebsz, G. Molnárka, *On the Separability of the H1/2 Norm in Finite Element Spaces,* Hungarian Electronic Journal, Vol. 1. (http://hej.szif.hu) (1999).

[11] B. Kiss, A. Krebsz, K. Szalay, *On the Circulant Representation of the H1/2 Seminorm on Tetrahedral Domains,* In: H. G. Bock et al. (eds.): ENUMATH 97, pp. 397–404, World Scientific, Singapore (1998).

[12] B. Kiss, A. Krebsz, K. Szalay, *On the Saparability of the H1/2 Seminorm on Convex Polygonal Domains,* Report 1998/1, Szécenyi István College, Győr, Hungary (1998).

[13] B. N. Khoromskij, W. L. Wendland, *Spectrally Equivalent Preconditioners for Boundary Equations in Substructuring Technics,* East-West J. Num. Math. Vol 1. pp. 1–27 (1993).

[14] J. Nečas, *Les méthodes directes en théorie des équations elliptices,* Academia, Prague (1967).

[15] P. Oswald, *Multilevel Finite Element Approximation,* Teubner, Stuttgart, (1994).

[16] O. Steinbach, W. L. Wendland, *The Construction of Efficient Preconditioners in the Boundary Element Method,* Preprint 97/29, University of Stuttgart, (1997).