On the Separability of the $H^{1/2}$ Seminorm on Convex Polygonal Domains

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Abstract

In this paper a generalization of the separability property of the $H^{1/2}$ seminorm is given for convex polygonal domains. Using this property, the $H^{1/2}$ seminorm on the surfaces of three-dimensional bounded domains can be represented as a simple circulant sparse matrix, which contains only $O(N\log(N))$ nonzero entries, where N denotes the number of unknowns.

1 Introduction

Let

$$\begin{aligned} \Omega &= \Big\{(x,y)\in R^2 \mid a_1\le x\le a_2, \quad \varphi_1(x)\le y\le \varphi_2(x)\Big\} \qquad (1)\\ &= \Big\{(x,y)\in R^2 \mid b_1\le y\le b_2, \quad \psi_1(y)\le x\le \psi_2(y)\Big\} \end{aligned}$$

be a given bounded domain in R^2. Then the $H^{1/2}$ seminorm, and the 'partial' seminorms $H_1^{1/2}$, $H_2^{1/2}$ are usually defined on Ω by the formulas [13, 5, 10, 11]

$$\mid f \mid^2_{H^{1/2}(\Omega)} = \int_\Omega\int_\Omega \frac{\mid f(x_1,y_1)-f(x_2,y_2)\mid^2}{\mid (x_1-x_2)^2+(y_1-y_2)^2\mid^{3/2}} dy_2dx_2dy_1dx_1, \qquad (2)$$

and

$$\mid f \mid^2_{H_1^{1/2}(\Omega)} = \int_{a_1}^{a_2}\int_{\varphi_1(x)}^{\varphi_2(x)}\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\mid f(x,y_1)-f(x,y_2)\mid^2}{\mid y_1-y_2\mid^2} dy_2dy_1dx, \qquad (3)$$

$$\mid f \mid^2_{H_2^{1/2}(\Omega)} = \int_{b_1}^{b_2}\int_{\psi_1(y)}^{\psi_2(y)}\int_{\psi_1(y)}^{\psi_2(y)} \frac{\mid f(x_1,y)-f(x_2,y)\mid^2}{\mid x_1-x_2\mid^2} dx_2dx_1dy, \qquad (4)$$

respectively. The separability property of the $H^{1/2}$ seminorm means, that it is spectrally equivalent to the sum of the 'partial' seminorms, i. e.

$$C_1\sum_{i=1}^{2}\mid f\mid^2_{H_i^{1/2}(\Omega)} \quad\le\quad \mid f\mid^2_{H^{1/2}(\Omega)} \quad\le\quad C_2\sum_{i=1}^{2}\mid f\mid^2_{H_i^{1/2}(\Omega)},$$

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where C_1 and C_2 are positive constants independent of f.

This property was proved at first in the case of hypercubes (cf. Lemma 5.3 in [13] or [5]). A different proof for general rectangular domains and its discrete equivalent in the space of bilinear finite elements has beeb given in [10]. The generalization of the property to triangular domains and its discrete equivalent in the space of linear finite elements has been discussed in [11]. The purpose of this paper to show that this property of the $H^{1/2}$ seminorm remains valid in the case of convex polygonal domains as well.

The matrix representations of the $H^{1/2}$ seminorm are efficient preconditioners for elliptic problems and boundary integral equations of first kind. Numerous papers are devoted to this topic, see for example [1, 3, 4, 5, 6, 7, 8, 12, 15].

By the use of the separability property, the $H^{1/2}$ seminorm can be represented as a sum of one-dimensional seminorms in the finite element spaces. Hence the $H^{1/2}$ seminorm on the surfaces of three-dimensional bounded domains can be represented as a simple sparse circulant matrix, which contains only $O(N\log(N))$ nonzero entries, where N denotes the number of unknowns. The construction of this matrix representation and its application as a Schur complement preconditioner in the case of brick shaped and tetrahedral domains are discussed in [10] and [11], respectively.

The matrix representation of the $H^{1/2}$ seminorm on the boundary of convex polyhedral domains, which is based on the separability property on convex domains, will be discussed in a separate paper.

2 The Separability Property

We prove the separabilty property of the $H^{1/2}$ seminorm in the case of the special convex polygonal domain

$$\Omega = \left\{(x,y)\in R^2 \mid 0\le x\le a_3, \quad \varphi_1(x)\le y\le \varphi_2(x)\right\}$$
$$= \left\{(x,y)\in R^2 \mid b_5\le y\le b_2, \quad \psi_1(y)\le x\le \psi_2(y)\right\},$$

which is defined by the partition (see Figure 1.)

$$\Omega = \left(\cup_{i=1}^{6} Q_i\right)\cup\left(\cup_{i=1}^{4} S_i\right), \tag{5}$$

where

$$Q_1=[0,a_1]\times[0,b_1], Q_2=[a_1,a_2]\times[0,b_2], Q_3=[a_2,a_3]\times[0,b_3],$$
$$Q_4=[0,a_4]\times[b_4,0], Q_5=[a_4,a_5]\times[b_5,0], Q_6=[a_5,a_6]\times[b_6,0],$$
$$S_1=\left\{(x,y)\in R^2 \mid 0\le x\le a_1, \quad b_1\le y\le \varphi_{21}(x)\right\},$$
$$S_2=\left\{(x,y)\in R^2 \mid a_2\le x\le a_3, \quad b_3\le y\le \varphi_{22}(x)\right\},$$
$$S_3=\left\{(x,y)\in R^2 \mid 0\le x\le a_4, \quad \varphi_{11}(x)\le y\le b_4\right\},$$
$$S_4=\left\{(x,y)\in R^2 \mid a_5\le x\le a_6, \quad \varphi_{12}(x)\le y\le b_6\right\},$$
$$0\le a_1\le a_2\le a_3, \quad 0\le a_4\le a_5\le a_3, \quad 0\le b_1,b_3\le b_2, \quad b_5\le b_4,b_6\le 0,$$

moreover φ_{12} and φ_{21} are strictly monotone increasing, φ_{11} and φ_{22} are strictly monotone decreasing piecewise linear continuous functions, such that

$$\varphi_{21}(0)=b_1, \quad \varphi_{21}(a_1)=b_2, \quad \varphi_{22}(a_2)=b_2, \quad \varphi_{22}(a_3)=b_3,$$
$$\varphi_{11}(0)=b_4, \quad \varphi_{11}(a_4)=b_5, \quad \varphi_{12}(a_5)=b_5, \quad \varphi_{12}(a_3)=b_6.$$

Remark 2.1 *Each convex polygonal domain can be written into this form if the direction x is chosen as the direction of one of the longest diagonals.*

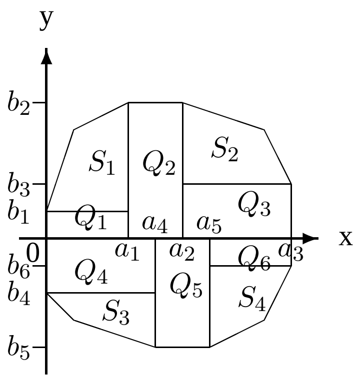

Figure 1.

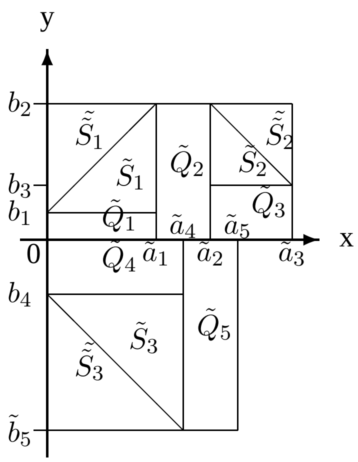

Figure 2.

A generalization of the separability property of the $H^{1/2}$ seminorm on this convex polygonal domain Ω can be formulated as follows:

Theorem 2.2

$$C_{11} \sum_{i=1}^{2} \mid f \mid^2_{H_i^{1/2}(\Omega)} \quad \leq \quad \mid f \mid^2_{H^{1/2}(\Omega)} \quad \leq \quad C_{12} \sum_{i=1}^{2} \mid f \mid^2_{H_i^{1/2}(\Omega)}, \tag{6}$$

for all $f \in H^{1/2}(\Omega)$, where C_{11} and C_{12} are positive constants independent of f.

The proof is based on a geometry simplification step and two extension steps for the extension of the simplified subdomains to rectangular domains.

We apply the following type of transformation to the simplification of the geometry of the subdomains of Ω:

Definition 2.3 (The transformation of the domain Ω_1) *Let*

$$\Omega_1 = \Omega \setminus (Q_6 \cup S_4)\,. \tag{7}$$

and let us define the transformation $T : \Omega_1 \to R^2$ on Ω_1 by the formula

$$T(x,y) = (T_1(x), T_2(y)), \quad \forall (x,y) \in \Omega_1, \tag{8}$$

where

$$T_1(x) = \begin{cases} \varphi_{21}(x) - b_1 & \text{if } x \in [0, a_1] \\ b_2 - b_1 + x - a_1 & \text{if } x \in [a_1, a_2] \\ b_2 - b_1 + a_2 - a_1 + b_2 - \varphi_{22}(x) & \text{if } x \in [a_2, a_3] \end{cases}, \tag{9}$$

and

$$T_2(y) = \begin{cases} b_4 - T_1(\varphi_{11}^{-1}(y)) & \text{if } y \in [b_5, b_4] \\ y & \text{if } y \in [b_4, b_2] \end{cases}. \tag{10}$$

Moreover, let $\tilde{\Omega}_1$ denote the range of T (see Figure 2.), that is

$$\tilde{\Omega}_1 = T(\Omega_1), \tag{11}$$

and let

$$\tilde{Q}_i = T(Q_i), \quad (i = 1,2,3,4,5), \quad \tilde{S}_i = T(S_i), \quad (i = 1,2,3). \tag{12}$$

The transformation T possesses the following properties:

Lemma 2.4 *The tranformation T is strictly monotone increasing and hence invertible. Moreover T is piecewise continuously differentiable and there exist positive constants C_{31} and C_{32} such that*

$$C_{31} \quad \leq \quad |\ T_1'(x)\ |, |\ T_2'(y)\ | \quad \leq \quad C_{32} \tag{13}$$

almost everywhere.

The transformed domain $\tilde{\Omega}_1$ is a union of rectangles and isosceales rectangular triangles, since

$$\tilde{\Omega}_1 = \left(\cup_{i=1}^5 \tilde{Q}_i \right) \cup \left(\cup_{i=1}^3 \tilde{S}_i \right), \tag{14}$$

where

$$\tilde{Q}_1 = [0, \tilde{a}_1] \times [0, b_1], \tilde{Q}_2 = [\tilde{a}_1, \tilde{a}_2] \times [0, b_2], \tilde{Q}_3 = [\tilde{a}_2, \tilde{a}_3] \times [0, b_3],$$
$$\tilde{Q}_4 = [0, \tilde{a}_4] \times [b_4, 0], \tilde{Q}_5 = [\tilde{a}_4, \tilde{a}_5] \times [\tilde{b}_5, 0],$$
$$\tilde{S}_1 = \left\{ (\tilde{x}, \tilde{y}) \in R^2 \mid 0 \leq \tilde{x} \leq \tilde{a}_1, \quad b_1 \leq \tilde{y} \leq b_1 + \tilde{x} \right\},$$
$$\tilde{S}_2 = \left\{ (\tilde{x}, \tilde{y}) \in R^2 \mid \tilde{a}_2 \leq \tilde{x} \leq \tilde{a}_3, \quad b_3 \leq \tilde{y} \leq b_3 + \tilde{a}_3 - \tilde{x} \right\},$$
$$\tilde{S}_3 = \left\{ (\tilde{x}, \tilde{y}) \in R^2 \mid 0 \leq \tilde{x} \leq \tilde{a}_4, \quad b_4 - \tilde{x} \leq \tilde{y} \leq b_4 \right\},$$

and $\tilde{a}_i = T_1(a_i)$ and $\tilde{b}_i = T_2(b_i)$.

Proof: The strictly monotone increasing property of T is a straightforward consequence of the construction. The piecewise differentiability of T follows from the piecewise linearity of the functions φ_{ij} $(i, j = 1, 2)$. The derivatives $T_1'(x)$ and $T_2'(y)$ can be expressed as follows:

$$T_1'(x) = \begin{cases} \varphi_{21}'(x) & \text{if } x \in [0, a_1] \\ 1 & \text{if } x \in [a_1, a_2] \\ -\varphi_{22}'(x) & \text{if } x \in [a_2, a_3] \end{cases},$$

and

$$T_2'(y) = \begin{cases} -T_1'(\varphi_{11}^{-1}(y)) \cdot \frac{1}{\varphi_{11}'(\varphi_{11}^{-1}(y))} & \text{if } y \in [b_5, b_4] \\ 1 & \text{if } y \in [b_4, b_2] \end{cases}.$$

Due to the strictly monotonicity and piecewise linearity of the functions φ_{ij} there exist positive constants $0 < C_1 \leq 1$ and $1 \leq C_2$ such that

$$C_1 \leq |\ \varphi_{ij}'(x)\ | \leq C_2$$

almost everywhere. Hence

$$C_1 \leq \ |\ \frac{\partial T_1(x)}{\partial x}\ |\ \leq C_2 \text{ and } \frac{C_1}{C_2} \leq \ |\ \frac{\partial T_2(y)}{\partial y}\ |\ \leq \frac{C_2}{C_1}$$

almost everywhere, and so the inequality of the lemma holds with the choice $C_{31} = \frac{C_1}{C_2}$ and $C_{32} = \frac{C_2}{C_1}$.

The formulas regarding the sets $\tilde{Q}_i$ and $\tilde{S}_i$ can be verified by simple calculation. ∎

Lemma 2.5 *There exist positive constants C_{41} and C_{42} independent of f such that*

$$C_{41} \mid f \mid^2_{H^{1/2}(\Omega_1)} \leq \mid \tilde{f} \mid^2_{H^{1/2}(\tilde{\Omega}_1)} \leq C_{42} \mid f \mid^2_{H^{1/2}(\Omega_1)}, \tag{15}$$

for all $f \in H^{1/2}(\Omega_1)$, and

$$C_{41} \mid f \mid^2_{H_i^{1/2}(\Omega_1)} \leq \mid \tilde{f} \mid^2_{H_i^{1/2}(\tilde{\Omega}_1)} \leq C_{42} \mid f \mid^2_{H_i^{1/2}(\Omega_1)}, \tag{16}$$

for all $f \in H_i^{1/2}(\Omega_1)$ $(i = 1, 2)$, where

$$\tilde{f}(\tilde{x}, \tilde{y}) = f(T_1^{-1}(\tilde{x}), T_2^{-1}(\tilde{y})), \quad \forall (\tilde{x}, \tilde{y}) \in \tilde{\Omega}_1. \tag{17}$$

Proof: We prove only the first inequality. The second one can be verified analogously. Let us consider the identity

$$\int_{\tilde{\Omega}_1}\int_{\tilde{\Omega}_1}\frac{\mid \tilde{f}(\tilde{x}_1,\tilde{y}_1)-\tilde{f}(\tilde{x}_2,\tilde{y}_2)\mid^2}{\mid (\tilde{x}_1-\tilde{x}_2)^2+(\tilde{y}_1-\tilde{y}_2)^2\mid^{3/2}}d\tilde{y}_2 d\tilde{x}_2 d\tilde{y}_1 d\tilde{x}_1 =$$

$$\int_{\Omega_1}\int_{\Omega_1}\frac{\mid f(x_1,y_1)-f(x_2,y_2)\mid^2}{\mid (T_1(x_1)-T_1(x_2))^2+(T_2(y_1)-T_2(y_2))^2\mid^{3/2}}\times$$

$$\times\left|\frac{\partial T(x_2,y_2)}{\partial(x_2,y_2)}\right|\cdot\left|\frac{\partial T(x_1,y_1)}{\partial(x_1,y_1)}\right| dy_2 dx_2 dy_1 dx_1,$$

where

$$\left|\frac{\partial T(x,y)}{\partial(x,y)}\right| = \begin{vmatrix} T_1'(x) & 0 \\ 0 & T_2'(y)\end{vmatrix} = |T_1'(x)|\cdot|T_2'(y)|$$

is the absolute value of the Jacobi determinant of T.

Applying Lemma 2.3 and the Lagrangian mean value theorem we get

$$C_{31}^3 \mid (x_1-x_2)^2+(y_1-y_2)^2\mid^{3/2}\le$$

$$\le \mid (T_1(x_1)-T_1(x_2))^2+(T_2(y_1)-T_2(y_2))^2\mid^{3/2}\le$$

$$\le C_{32}^3\mid (x_1-x_2)^2+(y_1-y_2)^2\mid^{3/2}$$

and

$$C_{31}^2\le\left|\frac{\partial T(x,y)}{\partial(x,y)}\right|\le C_{32}^2.$$

Hence

$$\frac{C_{31}^4}{C_{32}^3}\mid f\mid^2_{H^{1/2}(\Omega_1)}\le\mid\tilde{f}\mid^2_{H^{1/2}(\tilde{\Omega}_1)}\le\frac{C_{32}^4}{C_{31}^3}\mid f\mid^2_{H^{1/2}(\Omega_1)}$$

and so the lemma holds with the choice $C_{41}=\frac{C_{31}^4}{C_{32}^3}$ and $C_{42}=\frac{C_{32}^4}{C_{31}^3}$. ■

Definition 2.6 (An extension from $\tilde{\Omega}_1$ to an L-shaped domain) *Let*

$$\tilde{\tilde{\Omega}}_1=\tilde{\Omega}_1\cup\tilde{\tilde{S}}_1\cup\tilde{\tilde{S}}_2\cup\tilde{\tilde{S}}_3, \tag{18}$$

(see Figure 3.) where

$$\tilde{\tilde{S}}_1=\left\{(\tilde{x},\tilde{y})\in R^2\mid 0\le\tilde{x}\le\tilde{a}_1,\quad b_1+\tilde{x}\le\tilde{y}\le b_2\right\},$$

$$\tilde{\tilde{S}}_2=\left\{(\tilde{x},\tilde{y})\in R^2\mid \tilde{a}_2\le x\le\tilde{a}_3,\quad b_2-\tilde{x}+\tilde{a}_2\le\tilde{y}\le b_2\right\},$$

$$\tilde{\tilde{S}}_3=\left\{(\tilde{x},\tilde{y})\in R^2\mid 0\le\tilde{x}\le\tilde{a}_4,\quad \tilde{b}_5\le y\le b_4-\tilde{x}\right\}.$$

Moreover let

$$\tilde{\tilde{f}}(\tilde{x},\tilde{y})=$$

$$\begin{cases}\tilde{f}(\tilde{x},\tilde{y}) & \textit{if } (\tilde{x},\tilde{y})\in\tilde{\Omega}_1\\ \tilde{f}(\tilde{y}-b_1,\tilde{x}+b_1) & \textit{if } (\tilde{x},\tilde{y})\in\tilde{\tilde{S}}_1\\ \tilde{f}(-\tilde{y}+b_3+\tilde{a}_3,-\tilde{x}+\tilde{a}_2+b_2) & \textit{if } (\tilde{x},\tilde{y})\in\tilde{\tilde{S}}_2\\ \tilde{f}(-\tilde{y}+b_4,-\tilde{x}+b_4) & \textit{if } (\tilde{x},\tilde{y})\in\tilde{\tilde{S}}_3\end{cases}, \tag{19}$$

$$\forall\tilde{f}\in H^{1/2}(\tilde{\Omega}_1)\cup H_1^{1/2}(\tilde{\Omega}_1)\cup H_2^{1/2}(\tilde{\Omega}_1).$$

Some properties of this extension:

Lemma 2.7

$$| \tilde{\tilde{f}} |^2_{H^{1/2}(\tilde{\tilde{\Omega}}_1)} \leq 4 \mid \tilde{f} |^2_{H^{1/2}(\tilde{\Omega}_1)}, \tag{20}$$

for all $\tilde{f} \in H^{1/2}(\tilde{\Omega}_1)$*, and*

$$| \tilde{\tilde{f}} |^2_{H_i^{1/2}(\tilde{\tilde{\Omega}}_1)} \leq C_{61} \sum_{j=1}^{2} | \tilde{f} |^2_{H_j^{1/2}(\tilde{\Omega}_1)}, \quad (i = 1, 2), \tag{21}$$

for all $\tilde{f} \in H_1^{1/2}(\tilde{\Omega}_1) \cap H_2^{1/2}(\tilde{\Omega}_1)$*, where* C_{61} *is a positive constant independent of* $\tilde{f}$*.*

The proof of this lemma is based on the covering of $\tilde{\Omega}_1$ by special shaped domains (see Figure 3.-7.). The properties of the seminorms regarding these special shaped domains are discussed now.

Lemma 2.8 *Let* $Q = [0, a] \times [0, b]$ *where* $a, b > 0$*. Then*

$$2^{-3} \left(1 - \frac{max\{a^2, b^2\}}{(a+b)^2}\right) \sum_{i=1}^{2} | g |^2_{H_i^{1/2}(Q)} \leq | g |^2_{H^{1/2}(Q)} \leq 2^{5/2} \sum_{i=1}^{2} | g |^2_{H_i^{1/2}(Q)}, \tag{22}$$

for all $g \in H^{1/2}(Q)$.

Lemma 2.9 *Let* $c, d \in (0, \infty)$ *then*

$$\frac{1}{2}(1 - \frac{d^2}{(c+d)^2})\frac{1}{y^2} \leq \int_0^c \frac{1}{(y + | t - x |)^3} dt \leq \frac{1}{y^2}, \quad x \in [0, c], y \in (0, d] \tag{23}$$

The proof of Lemmata 2.8 and 2.9 can be found in [10].

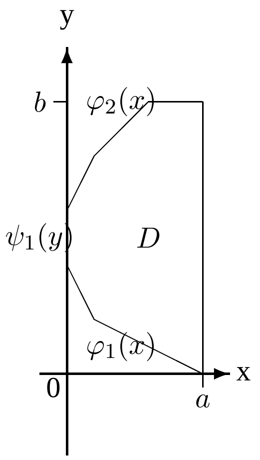

Figure 3.

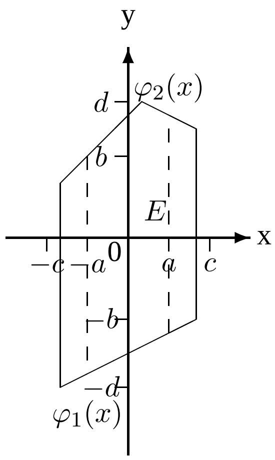

Figure 4.

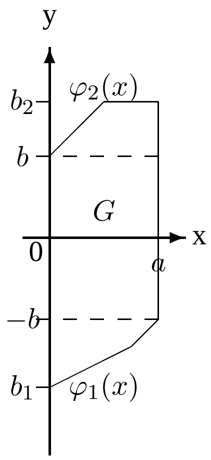

Figure 5.

Lemma 2.10 *Let*

$$D = \left\{(x, y) \in R^2 \mid \quad 0 \leq x \leq a, \quad 0 \leq \varphi_1(x) \leq y \leq \varphi_2(x)\right\}$$
$$= \left\{(x, y) \in R^2 \mid \quad 0 \leq y \leq b, \quad 0 \leq \psi_1(y) \leq x \leq a\right\}$$

be a convex domain (see Figure 3.) where $a, b > 0$*,* φ_1 *and* φ_2 *are monotone decreasing and monotone increasing continuous functions, respectively.*

Then

$$| g |^2_{H^{1/2}(D)} \leq 2^{7/2} \left(| g |^2_{H_1^{1/2}(D)} + | g |^2_{H_2^{1/2}(D)}\right) \tag{24}$$

for all $g \in H_1^{1/2}(D) \cap H_2^{1/2}(D)$.

Proof: A straightforward computation gives

$$| g |^2_{H^{1/2}(D)} =$$

$$2\int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{x_1} \int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{| g(x_1,y_1) - g(x_2,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_2 dx_2 dy_1 dx_1.$$

Since $g(x_1, y_2)$ is well defined when $x_2 \leq x_1$, by the use of the inequality

$$(c+d)^2 \leq 2c^2 + 2d^2$$

we get

$$| g |^2_{H^{1/2}(D)} \leq 4I_1 + 4I_2,$$

where

$$I_1 = \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{x_1} \int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{| g(x_1,y_1) - g(x_1,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_2 dx_2 dy_1 dx_1,$$

and

$$I_2 = \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{x_1} \int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{| g(x_1,y_2) - g(x_2,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_2 dx_2 dy_1 dx_1.$$

The estimation of the integral I_1:

Take $\varphi_1(x_1) \geq \varphi_1(x_2)$ and $\varphi_2(x_1) \leq \varphi_2(x_2)$ into account to see that

$$I_1 \leq \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{x_1} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \frac{| g(x_1,y_1) - g(x_1,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_2 dx_2 dy_1 dx_1 = I_{11}.$$

Due to $x_2 \leq a$ and the inequality $\frac{1}{(c^2+d^2)^{3/2}} \leq 2^{3/2} \frac{1}{(c+d)^3}, \quad \forall c, d \in (0, \infty)$,

$$I_{11} \leq \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{a} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \frac{| g(x_1,y_1) - g(x_1,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_2 dx_2 dy_1 dx_1 \leq$$

$$\leq 2^{3/2} \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{a} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \frac{| g(x_1,y_1) - g(x_1,y_2) |^2}{(| x_1 - x_2 | + | y_1 - y_2 |)^3} dy_2 dx_2 dy_1 dx_1 =$$

$$= 2^{3/2} \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{a} \frac{| g(x_1,y_1) - g(x_1,y_2) |^2}{(| x_1 - x_2 | + | y_1 - y_2 |)^3} dx_2 dy_2 dy_1 dx_1 = I_{12}.$$

Apply Lemma 2.9 $c = d = a$, $y = | y_1 - y_2 |$, $t = x_2$, $x = x_1$ to obtain

$$I_{12} \leq 2^{3/2} \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \frac{| g(x_1,y_1) - g(x_1,y_2) |^2}{| y_1 - y_2 |^2} dy_2 dy_1 dx_1 =$$

$$= 2^{3/2} | g |^2_{H_1^{1/2}(D)}.$$

The integral I_2 can be estimated similarly:

$$I_2 = \int_0^a \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_{\psi_1(y_2)}^{x_1} \frac{| g(x_1,y_2) - g(x_2,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dx_2 dy_2 dy_1 dx_1 \leq$$

$$\leq \int_0^a \int_0^b \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_{\psi_1(y_2)}^{x_1} \frac{| g(x_1,y_2) - g(x_2,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dx_2 dy_2 dy_1 dx_1 =$$

$$= \int_0^b \int_{\psi_1(y_2)}^{a} \int_{\psi_1(y_2)}^{x_1} \int_0^{b} \frac{| g(x_1,y_2) - g(x_2,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_1 dx_2 dx_1 dy_2 \leq$$

$$\leq \int_0^b \int_{\psi_1(y_2)}^a \int_{\psi_1(y_2)}^a \int_0^b \frac{| g(x_1,y_2) - g(x_2,y_2) |^2}{(| x_1 - x_2 |^2 + | y_1 - y_2 |^2)^{3/2}} dy_1 dx_2 dx_1 dy_2 \leq$$

$$\leq 2^{3/2} \int_0^b \int_{\psi_1(y_2)}^a \int_{\psi_1(y_2)}^a \frac{| g(x_1,y_2) - g(x_2,y_2) |^2}{| x_1 - x_2 |^2} dx_2 dx_1 dy_2 = 2^{3/2} | g |^2_{H_2^{1/2}(D)}.$$

This completes the proof. ∎

Definition 2.11 *In the following we use the notation*

$$| h |^2_{H^{1/2}[T_1,T_2]} = \int_{T_1} \int_{T_2} \frac{| h(x_1,y_1) - h(x_2,y_2) |^2}{| (x_1 - x_2)^2 + (y_1 - y_2)^2 |^{3/2}} dy_2 dx_2 dy_1 dx_1. \tag{25}$$

Lemma 2.12 *Let*

$$E = \left\{ (x,y) \in R^2 \mid \quad a_1 \leq x \leq a_2, \quad \varphi_1(x) \leq y \leq \varphi_2(x) \right\}$$

be a convex domain (see Figure 4.) where φ_1 and φ_2 are continuous functions. Assume that there exist positive constants $0 < a, b, c, d$ such that

$$[-a, a] \times [-b, b] \subseteq E \subseteq [-c, c] \times [-d, d]$$

and let

$$E_- = \{(x,y) \in E \mid \quad a_1 \leq x \leq 0\},$$
$$E_0 = \{(x,y) \in E \mid \quad -a \leq x \leq a\},$$
$$E_+ = \{(x,y) \in E \mid \quad 0 \leq x \leq a_2\}.$$

Then

$$C_{121} | g |^2_{H^{1/2}(E)} \leq \left(| g |^2_{H^{1/2}(E_- \cup E_0)} + | g |^2_{H^{1/2}(E_+ \cup E_0)} \right) \leq C_{122} | g |^2_{H^{1/2}(E)}, \tag{26}$$

for all $g \in H^{1/2}(E)$, where C_{121}, C_{122} are a positive constants independent of g.

Proof: The second inequality obviously holds with $C_{122} = 2$, so it is enough to prove the first one. Take $E = (E_- \cup E_0) \cup (E_+ \setminus E_0) = (E_+ \cup E_0) \cup (E_- \setminus E_0)$ into account

$$| g |^2_{H^{1/2}(E)} \leq | g |^2_{H^{1/2}(E_- \cup E_0)} + | g |^2_{H^{1/2}(E_+ \cup E_0)} +$$

$$+ | g |^2_{H^{1/2}[E_- \setminus E_0, E_+]} + | g |^2_{H^{1/2}[E_+ \setminus E_0, E_-]}.$$

Applying the inequality

$$(h_1 + h_2 + h_3)^2 \leq 3 \cdot \left(h_1^2 + h_2^2 + h_3^2 \right)$$

we get

$$| g |^2_{H^{1/2}[E_- \setminus E_0, E_+]} \leq 3 \cdot (I_1 + I_2 + I_3),$$

where

$$I_1 = \int_{a_1}^{-a} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{a_2} \int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{| g(\frac{a}{c}x_1, \frac{b}{d}y_1) - g(x_2,y_2) |^2}{| (x_1 - x_2)^2 + (y_1 - y_2)^2 |^{3/2}} dy_2 dx_2 dy_1 dx_1,$$

$$I_2 = \int_{a_1}^{-a} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{a_2} \int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{| g(\frac{a}{c}x_1, \frac{b}{d}y_1) - g(\frac{a}{c}x_2, \frac{b}{d}y_2) |^2}{| (x_1 - x_2)^2 + (y_1 - y_2)^2 |^{3/2}} dy_2 dx_2 dy_1 dx_1,$$

and

$$I_3 = \int_{a_1}^{-a} \int_{\varphi_1(x_1)}^{\varphi_2(x_1)} \int_0^{a_2} \int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{| g(x_1,y_1) - g(\frac{a}{c}x_2, \frac{b}{d}y_2) |^2}{| (x_1 - x_2)^2 + (y_1 - y_2)^2 |^{3/2}} dy_2 dx_2 dy_1 dx_1.$$

The integral I_1 can be estimated as follows

$$I_1 \le \int_{a_1}^{-a}\int_{\varphi_1(x_1)}^{\varphi_2(x_1)}\int_0^{a_2}\int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{\mid g(\frac{a}{c}x_1, \frac{b}{d}y_1) - g(x_2,y_2)\mid^2}{\mid (\frac{a}{c}x_1 - x_2)^2 + (\frac{b}{d}y_1 - y_2)^2 \mid^{3/2}} dy_2 dx_2 dy_1 dx_1 =$$

$$= \left(\frac{c\cdot d}{a\cdot b}\right)\cdot \int_{\frac{a}{c}a_1}^{-\frac{a}{c}a}\int_{\frac{b}{d}\varphi_1(\frac{c}{a}\tilde{x}_1)}^{\frac{b}{d}\varphi_2(\frac{c}{a}\tilde{x}_1)}\int_0^{a_2}\int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{\mid g(\tilde{x}_1,\tilde{y}_1) - g(x_2,y_2)\mid^2}{\mid (\tilde{x}_1 - x_2)^2 + (\tilde{y}_1 - y_2)^2 \mid^{3/2}} dy_2 dx_2 d\tilde{y}_1 d\tilde{x}_1 \le$$

$$\le \left(\frac{c\cdot d}{a\cdot b}\right)\cdot \int_{-a}^{a_2}\int_{\varphi_1(\tilde{x}_1)}^{\varphi_2(\tilde{x}_1)}\int_{-a}^{a_2}\int_{\varphi_1(x_2)}^{\varphi_2(x_2)} \frac{\mid g(\tilde{x}_1,\tilde{y}_1) - g(x_2,y_2)\mid^2}{\mid (\tilde{x}_1 - x_2)^2 + (\tilde{y}_1 - y_2)^2 \mid^{3/2}} dy_2 dx_2 d\tilde{y}_1 d\tilde{x}_1 =$$

$$= \left(\frac{c\cdot d}{a\cdot b}\right)\cdot \mid g \mid^2_{H^{1/2}(E_+\cup E_0)},$$

where we have used the variable substitution $\tilde{x}_1 = \frac{a}{c}x_1$ and $\tilde{y}_1 = \frac{b}{d}y_1$. Performing similar estimations we get

$$I_2 \le \left(\frac{c\cdot d}{a\cdot b}\right)^2 \cdot \mid g \mid^2_{H^{1/2}(E_+\cup E_0)} \text{ and } I_3 \le \left(\frac{c\cdot d}{a\cdot b}\right)\cdot \mid g \mid^2_{H^{1/2}(E_-\cup E_0)}.$$

Hence

$$\mid g \mid^2_{H^{1/2}[E_-\setminus E_0, E_+]} \le 3\cdot \left(\frac{c\cdot d}{a\cdot b}\right)^2 \cdot \left(2\cdot \mid g \mid^2_{H^{1/2}(E_+\cup E_0)} + \mid g \mid^2_{H^{1/2}(E_-\cup E_0)}\right)$$

and similarly

$$\mid g \mid^2_{H^{1/2}[E_+\setminus E_0, E_-]} \le 3\cdot \left(\frac{c\cdot d}{a\cdot b}\right)^2 \cdot \left(2\cdot \mid g \mid^2_{H^{1/2}(E_-\cup E_0)} + \mid g \mid^2_{H^{1/2}(E_+\cup E_0)}\right).$$

Summing up the previous inequalities leads to

$$\mid g \mid^2_{H^{1/2}(E)} \le 10\cdot \left(\frac{c\cdot d}{a\cdot b}\right)^2 \cdot \left(\mid g \mid^2_{H^{1/2}(E_-\cup E_0)} + \mid g \mid^2_{H^{1/2}(E_+\cup E_0)}\right)$$

and so the first inequality of the lemma holds with $C_{121} = \frac{1}{10}\cdot\left(\frac{a\cdot b}{c\cdot d}\right)^2$. ■

Lemma 2.13 *Let*

$$G = \left\{(x,y) \in R^2 \mid \quad 0 \le x \le a, \quad b_1 \le \varphi_1(x) \le y \le \varphi_2(x) \le b_2\right\}$$

be a convex domain (see Figure 5.) where $a, b > 0$, $-b_1, b_2 \ge b$. Assume that φ_1 and φ_2 are monotone increasing continuous functions and

$$[0,a]\times[-b,b] \subset G.$$

Then

$$\mid g \mid^2_{H^{1/2}(G)} \le C_{131}\left(\mid g \mid^2_{H_1^{1/2}(G)} + \mid g \mid^2_{H_2^{1/2}(G)}\right) \tag{27}$$

for all $g \in H_1^{1/2}(G)\cap H_2^{1/2}(G)$, where C_{131} is a positive constant independent of g.

Proof: Let us introduce the subsets

$$G_- = \left\{(x,y)\in R^2 \mid \quad 0 \le x \le a, \quad \varphi_1(x) \le y \le 0\right\},$$

$$G_0 = [0,a]\times[-b,b],$$

and

$$G_+ = \left\{(x,y)\in R^2 \mid \quad 0 \le x \le a, \quad 0 \le y \le \varphi_2(x)\right\}$$

of G. Then by application of Lemma 2.12

$$| g |^2_{H^{1/2}(G)} \leq \frac{1}{C_{121}} \cdot \left(| g |^2_{H^{1/2}(G_- \cup G_0)} + | g |^2_{H^{1/2}(G_+ \cup G_0)} \right).$$

Since the sets $(G_- \cup G_0)$ and $(G_+ \cup G_0)$ satisfy the conditions of Lemma 2.10, we have

$$| g |^2_{H^{1/2}(G)} \leq$$

$$\leq \frac{1}{C_{121}} \cdot \left(2^{7/2} \cdot \sum_{i=1}^{2} | g |^2_{H_i^{1/2}(G_- \cup G_0)} + 2^{7/2} \cdot \sum_{i=1}^{2} | g |^2_{H_i^{1/2}(G_+ \cup G_0)} \right) \leq$$

$$\leq 2 \cdot 2^{7/2} \cdot \frac{1}{C_{121}} \cdot \sum_{i=1}^{2} | g |^2_{H_i^{1/2}(G)}$$

and so the lemma holds with $C_{131} = 2 \cdot 2^{7/2} \cdot \frac{1}{C_{121}}$. ■

Proof of Lemma 2.7:

The proof of the first inequality:

$$| \tilde{\tilde{f}} |^2_{H^{1/2}(\tilde{\tilde{\Omega}}_1)} = | \tilde{f} |^2_{H^{1/2}(\tilde{\Omega}_1)} + \sum_{i=1}^{3} | \tilde{\tilde{f}} |^2_{H^{1/2}(\tilde{\tilde{S}}_i)} +$$

$$+2 \sum_{i=1}^{3} | \tilde{\tilde{f}} |^2_{H^{1/2}[\tilde{\Omega}_1, \tilde{\tilde{S}}_i]} + 2 \sum_{i=1}^{2} \sum_{j=i+1}^{3} | \tilde{\tilde{f}} |^2_{H^{1/2}[\tilde{\tilde{S}}_i, \tilde{\tilde{S}}_j]} \leq$$

$$\leq | \tilde{f} |^2_{H^{1/2}(\tilde{\Omega}_1)} + \sum_{i=1}^{3} | \tilde{f} |^2_{H^{1/2}(\tilde{S}_i)} +$$

$$+2 \sum_{i=1}^{3} | \tilde{f} |^2_{H^{1/2}[\tilde{\Omega}_1, \tilde{S}_i]} + 2 \sum_{i=1}^{2} \sum_{j=i+1}^{3} | \tilde{f} |^2_{H^{1/2}[\tilde{S}_i, \tilde{S}_j]} \leq 4 \cdot | \tilde{f} |^2_{H^{1/2}(\tilde{\Omega}_1)} .$$

The proof of the second inequality:

We prove only the most complicated case, when $i = 1$ and $0 < \tilde{a}_1 \leq \tilde{a}_2 < \tilde{a}_4 \leq \tilde{a}_5 \leq \tilde{a}_3$. The remaining cases can be proved analogously. In the proof c_1, c_2 etc. denote positive constants independent of f. Since $\tilde{\Omega}_1 = H_1 \cup H_2$, where

$$H_1 = [0, \tilde{a}_3] \times [0, b_2], \text{ and } H_2 = [\tilde{a}_1, \tilde{a}_5] \times [\tilde{b}_5, b_2]$$

by the application of the first inequality of Lemma 2.8 and the previously proved inequality it follows that

$$| \tilde{\tilde{f}} |^2_{H_1^{1/2}(\tilde{\tilde{\Omega}}_1)} \leq | \tilde{\tilde{f}} |^2_{H_1^{1/2}(H_1)} + | \tilde{\tilde{f}} |^2_{H_1^{1/2}(H_2)} \leq$$

$$\leq c_1 \cdot \left(| \tilde{\tilde{f}} |^2_{H^{1/2}(H_1)} + | \tilde{\tilde{f}} |^2_{H^{1/2}(H_2)} \right) \leq$$

$$\leq 4 \cdot c_1 \left(| \tilde{f} |^2_{H^{1/2}(H_1 \cap \tilde{\Omega}_1)} + | \tilde{f} |^2_{H^{1/2}(H_2 \cap \tilde{\Omega}_1)} \right) = I_1.$$

The shape of the domain $H_1 \cap \tilde{\Omega}_1$ satisfies the hypotheses of Lemma 2.10 which hence can be applied directly to this domain. However the domain $H_2 \cap \tilde{\Omega}_1$ needs a covering by such domains which satisfy the hypotheses of Lemma 2.10 or 2.13. Such a covering (see Figure 5-7) can be done by the rectangles

$$H_{21} = [0, \tilde{a}_1] \times [\tilde{b}_5, b_2], \quad H_{22} = [\tilde{a}_2, \tilde{a}_2 + \frac{2}{3}(\tilde{a}_5 - \tilde{a}_2)] \times [\tilde{b}_5, b_2],$$

$$H_{23} = [\tilde{a}_2 + \frac{1}{3}(\tilde{a}_5 - \tilde{a}_2), \tilde{a}_5] \times [\tilde{b}_5, b_2], \quad H_{24} = [\frac{1}{2}\tilde{a}_1, \tilde{a}_2 + \frac{1}{3}(\tilde{a}_5 - \tilde{a}_2)] \times [b_4 - \frac{1}{2}\tilde{a}_1, b_2],$$

$$H_{25} = [\frac{1}{2}\tilde{a}_1, \tilde{a}_2 + \frac{1}{3}(\tilde{a}_5 - \tilde{a}_2)] \times [\tilde{b}_5, 0].$$

Then using Lemma 2.12 there follows

$$\mid \tilde{f} \mid^2_{H^{1/2}(H_2 \cap \tilde{\Omega}_1)} \leq c_2 \cdot \sum_{j=1}^{5} \mid \tilde{f} \mid^2_{H^{1/2}(H_{2j} \cap \tilde{\Omega}_1)}$$

and hence

$$I_1 \leq c_3 \cdot \left(\mid \tilde{f} \mid^2_{H^{1/2}(H_1 \cap \tilde{\Omega}_1)} + \sum_{j=1}^{5} \mid \tilde{f} \mid^2_{H^{1/2}(H_{2j} \cap \tilde{\Omega}_1)} \right) = I_2.$$

Applying Lemma 2.10 and 2.13 we get

$$I_2 \leq c_4 \cdot \left(\sum_{i=1}^{2} \mid \tilde{f} \mid^2_{H_i^{1/2}(H_1 \cap \tilde{\Omega}_1)} + \sum_{i=1}^{2} \sum_{j=1}^{5} \mid \tilde{f} \mid^2_{H_i^{1/2}(H_{2j} \cap \tilde{\Omega}_1)} \right) \leq c_5 \cdot \sum_{i=1}^{2} \mid \tilde{f} \mid^2_{H_i^{1/2}(\tilde{\Omega}_1)} .$$

The proof is complete. ■

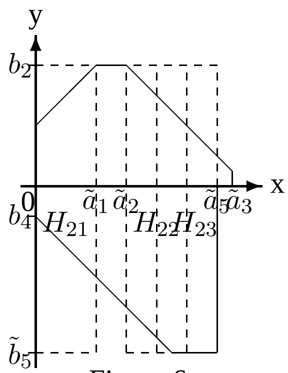

Figure 6.

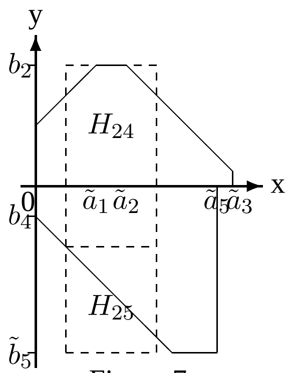

Figure 7.

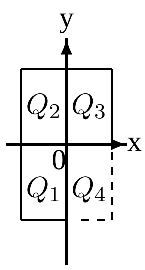

Figure 8.

Definition 2.14 (An extension from an L-shaped domain to a rectangle) *Let*

$$L = Q_1 \cup Q_2 \cup Q_3 \text{ and } \tilde{L} = L \cup Q_4, \tag{28}$$

(see Figure 8.) where

$$Q_1 = [-c_1, 0] \times [-d_1, 0], \quad Q_2 = [-c_1, 0] \times [0, d_2],$$

$$Q_3 = [0, c_2] \times [0, d_2], \quad Q_4 = [0, c_2] \times [-d_1, 0],$$

and $0 < c_1, c_2, d_1, d_2$.

Moreover let

$$\tilde{g}(x,y) = \begin{cases} g(x,y) & \text{if} \quad (x,y) \in L \\ g(-\frac{c_1}{c_2}x, y) + g(x, -\frac{d_2}{d_1}y) - g(-\frac{c_1}{c_2}x, -\frac{d_2}{d_1}y) & \text{if} \quad (x,y) \in Q_4 \end{cases}, \tag{29}$$

$$\forall g \in H^{1/2}(L) \cup H_1^{1/2}(L) \cup H_2^{1/2}(L).$$

Lemma 2.15 *There holds*

$$\mid \tilde{g} \mid^2_{H^{1/2}(\tilde{L})} \leq C_{131} \cdot \mid g \mid^2_{H^{1/2}(L)}, \tag{30}$$

for all $g \in H^{1/2}(L)$*, and*

$$\mid \tilde{g} \mid^2_{H_i^{1/2}(\tilde{L})} \leq C_{131} \cdot \mid g \mid^2_{H_i^{1/2}(L)}, \tag{31}$$

for all $g \in H_i^{1/2}(L)$ $(i = 1, 2)$*, where* C_{131} *is a positive constant independent of* g*.*

Proof: The proof is based on the application of the inequalities

$$(a_1 + a_2 + a_3)^2 \leq 3 \cdot \left(a_1^2 + a_2^2 + a_3^2\right)$$

and

$$(b_1 - b_2)^2 \leq (b_1 + b_2)^2, \quad b_1, b_2 \geq 0.$$

The proof of the first inequality:

Since

$$\mid \tilde{g} \mid^2_{H^{1/2}[Q_1,Q_4]} \leq 3 \cdot M \cdot \left(\mid g \mid^2_{H^{1/2}(Q_1)} + \mid g \mid^2_{H^{1/2}(Q_2)} + \mid g \mid^2_{H^{1/2}[Q_2,Q_3]}\right),$$

$$\mid \tilde{g} \mid^2_{H^{1/2}[Q_2,Q_4]} \leq 3 \cdot M \cdot \left(\mid g \mid^2_{H^{1/2}[Q_1,Q_2]} + \mid g \mid^2_{H^{1/2}(Q_2)} + \mid g \mid^2_{H^{1/2}[Q_2,Q_3]}\right),$$

$$\mid \tilde{g} \mid^2_{H^{1/2}[Q_3,Q_4]} \leq 3 \cdot M \cdot \left(\mid g \mid^2_{H^{1/2}[Q_1,Q_2]} + \mid g \mid^2_{H^{1/2}(Q_2)} + \mid g \mid^2_{H^{1/2}(Q_3)}\right),$$

$$\mid \tilde{g} \mid^2_{H^{1/2}(Q_4)} \leq 3 \cdot M \cdot \left(\mid g \mid^2_{H^{1/2}(Q_1)} + \mid g \mid^2_{H^{1/2}(Q_2)} + \mid g \mid^2_{H^{1/2}(Q_3)},\right)$$

where

$$M = \max\left\{1, \left(\frac{c_1}{c_2}\right)^4, \left(\frac{d_2}{d_1}\right)^4\right\}$$

we get

$$\mid \tilde{g} \mid^2_{H^{1/2}(\tilde{L})} = \mid g \mid^2_{H^{1/2}(L)} + \mid \tilde{g} \mid^2_{H^{1/2}(Q_4)} + 2\sum_{i=1}^{3} \mid \tilde{g} \mid^2_{H^{1/2}[Q_i,Q_4]} \leq \; 21 \cdot M \cdot \mid g \mid^2_{H^{1/2}(L)}.$$

The proof of the second inequality:

We prove only the $H_1^{1/2}$ case. Let us introduce the notations

$$\mid g \mid^2_{H_1^{1/2}[Q_1,Q_2]} = \int_{-c_1}^{0}\int_{-d_1}^{0}\int_{0}^{d_2} \frac{\mid g(x,y_1) - g(x,y_2) \mid^2}{\mid y_1 - y_2 \mid^2} dy_2 dy_1 dx$$

and

$$\mid \tilde{g} \mid^2_{H_1^{1/2}[Q_3,Q_4]} = \int_{0}^{c_2}\int_{0}^{d_2}\int_{-d_1}^{0} \frac{\mid g(x,y_1) - \tilde{g}(x,y_2) \mid^2}{\mid y_1 - y_2 \mid^2} dy_2 dy_1 dx.$$

Since

$$\mid \tilde{g} \mid^2_{H_1^{1/2}(Q_4)} \leq 3 \cdot M \cdot \left(\mid g \mid^2_{H_1^{1/2}(Q_1)} + \mid g \mid^2_{H_1^{1/2}(Q_2)} + \mid g \mid^2_{H_1^{1/2}(Q_3)}\right)$$

and

$$\mid \tilde{g} \mid^2_{H_1^{1/2}[Q_3,Q_4])} \leq 3 \cdot M \cdot \left(\mid g \mid^2_{H_1^{1/2}(Q_2)} + \mid g \mid^2_{H_1^{1/2}(Q_3)} + \mid g \mid^2_{H_1^{1/2}[Q_1,Q_2]}\right)$$

we get

$$\mid \tilde{g} \mid^2_{H_1^{1/2}(\tilde{L})} = \mid g \mid^2_{H_1^{1/2}(Q_1 \cup Q_2)} + \mid g \mid^2_{H_1^{1/2}(Q_3)} + \mid \tilde{g} \mid^2_{H_1^{1/2}(Q_4)} + 2 \cdot \mid \tilde{g} \mid^2_{H_1^{1/2}[Q_3,Q_4]} \leq$$

$$\leq \quad 10 \cdot M \cdot \mid g \mid^2_{H_1^{1/2}(L)}.$$

Hence the lemma holds with $C_{131} = 21 \cdot M$. ∎

Proof of Theorem 2.2.: We consider only the most complicated case when $0 < a_1 < a_4 < a_2 < a_5 < a_3$ and $b_5 < b_4, b_6 < 0 < b_1, b_3 < b_2$. The remaining cases can be proved analogously.

Let

$$\Omega_1 = S_1 \cup S_2 \cup S_3 \cup Q_2 \cup Q_5 \cup Q_1 \cup Q_3 \cup Q_4,$$
$$\Omega_2 = S_1 \cup S_2 \cup S_4 \cup Q_2 \cup Q_5 \cup Q_1 \cup Q_3 \cup Q_6,$$
$$\Omega_3 = S_1 \cup S_3 \cup S_4 \cup Q_2 \cup Q_5 \cup Q_1 \cup Q_4 \cup Q_6,$$
$$\Omega_4 = S_2 \cup S_3 \cup S_4 \cup Q_2 \cup Q_5 \cup Q_3 \cup Q_4 \cup Q_6.$$

Then

$$| f |^2_{H^{1/2}(\Omega)} \le \sum_{j=1}^{4} | f |^2_{H^{1/2}(\Omega_j)} \le 4 | f |^2_{H^{1/2}(\Omega)},$$

and

$$| f |^2_{H_i^{1/2}(\Omega)} \le \sum_{j=1}^{4} | f |^2_{H_i^{1/2}(\Omega_j)} \le 4 | f |^2_{H_i^{1/2}(\Omega)}, \quad (i = 1, 2).$$

Hence it is enough to show that the following inequalities hold with some positive constants c_1 and c_2:

$$c_1 \sum_{i=1}^{2} | f |^2_{H_i^{1/2}(\Omega_j)} \le | f |^2_{H^{1/2}(\Omega_j)} \le c_2 \sum_{i=1}^{2} | f |^2_{H_i^{1/2}(\Omega_j)}, \quad (j = 1, \ldots, 4).$$

We prove only the Ω_1 case. The proof is based on the sequential application of Lemmata 2.5,2.7,2.15 and 2.8, where c_1, c_2 etc. denote positive constants:

The proof of the first inequality:

$$\sum_{i=1}^{2} | f |^2_{H_i^{1/2}(\Omega_1)} \le c_3 \sum_{i=1}^{2} | \tilde{f} |^2_{H_i^{1/2}(\tilde{\Omega}_1)} \le c_4 \sum_{i=1}^{2} | \tilde{\tilde{f}} |^2_{H_i^{1/2}(\tilde{\tilde{\Omega}}_1)} \le$$

$$\le c_5 \sum_{i=1}^{2} | \tilde{\tilde{\tilde{f}}} |^2_{H_i^{1/2}(\tilde{\tilde{\tilde{\Omega}}}_1)} \le c_6 | \tilde{\tilde{\tilde{f}}} |^2_{H^{1/2}(\tilde{\tilde{\tilde{\Omega}}}_1)} \le c_7 | \tilde{\tilde{f}} |^2_{H^{1/2}(\tilde{\tilde{\Omega}}_1)} \le$$

$$\le c_8 | \tilde{f} |^2_{H^{1/2}(\tilde{\Omega}_1)} \le c_9 | f |^2_{H^{1/2}(\Omega_1)} .$$

The proof of the second inequality:

$$| f |^2_{H^{1/2}(\Omega_1)} \le c_{10} | \tilde{f} |^2_{H^{1/2}(\tilde{\Omega}_1)} \le c_{11} | \tilde{\tilde{f}} |^2_{H^{1/2}(\tilde{\tilde{\Omega}}_1)} \le$$

$$\le c_{12} | \tilde{\tilde{\tilde{f}}} |^2_{H^{1/2}(\tilde{\tilde{\tilde{\Omega}}}_1)} \le c_{13} \sum_{i=1}^{2} | \tilde{\tilde{\tilde{f}}} |^2_{H_i^{1/2}(\tilde{\tilde{\tilde{\Omega}}}_1)} \le c_{14} \sum_{i=1}^{2} | \tilde{\tilde{f}} |^2_{H_i^{1/2}(\tilde{\tilde{\Omega}}_1)} \le$$

$$\le c_{15} \sum_{i=1}^{2} | \tilde{f} |^2_{H_i^{1/2}(\tilde{\Omega}_1)} \le c_{16} \sum_{i=1}^{2} | f |^2_{H_i^{1/2}(\Omega_1)} .$$

■

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