Computer algorithms for fundamental domains of $\mathbb{E}^4$ space groups (to decagonal and icosahedral families)

László Ács

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Abstract

As an introduction we illustrate the D-V cells and fundamental domains of the plane group **p3** of $\mathbb{E}^2$ [IT].

Then we describe some algorithms for determining fundamental domains of 4-dimensional space groups, in general. Our example space groups belong to the most interesting decagonal and icosahedral families of $\mathbb{E}^4$. These are
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by [BBNWZ].

As new results we get interesting space tiling 4-polytopes which can also be visualized by computer.

1 Space groups in $\mathbb{E}^d$

An isometric transformation (briefly transform) α as element of a *d-dimensional space group* Γ can be described in any coordinate system with a $(d+1)\times(d+1)$ matrix $\alpha := \begin{pmatrix} \mathbf{A} & \mathbf{a} \\ 0 & 1 \end{pmatrix}$. We write $\alpha(\mathbf{a};\mathbf{A})$ as well. The $d\times d$ matrix $\mathbf{A}$ is called the *linear part* of α, the column vector $\mathbf{a}$ is its *translational part.* For a space group Γ, all these are specially expressed in a lattice coordinate system $(O,\mathbf{e}_1,\dots,\mathbf{e}_d)$ with the symmetric *Gramian*

(1.1)
$$(\mathbf{g}_{ij})=(\langle\mathbf{e}_i;\mathbf{e}_j\rangle) = \begin{pmatrix} \langle\mathbf{e}_1;\mathbf{e}_1\rangle & \langle\mathbf{e}_1;\mathbf{e}_2\rangle & \dots & \langle\mathbf{e}_1;\mathbf{e}_n\rangle \\ & \langle\mathbf{e}_2;\mathbf{e}_2\rangle & \dots & \langle\mathbf{e}_2;\mathbf{e}_n\rangle \\ & & & \vdots \\ & & & \langle\mathbf{e}_n;\mathbf{e}_n\rangle \end{pmatrix}.$$

Then α associates each point X with its image $\alpha X =: Y$ by $(d+1)$-row-column multiplication as usual:

(1.2)
$$\begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} := \begin{pmatrix} \mathbf{A} & \mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{Ax}+\mathbf{a} \\ 1 \end{pmatrix};\ X\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, Y\begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix}$$

are introduced. The lattice $\mathbf{L}$ is the *integral linear combinations* of the basis vectors $(\mathbf{e}_1,\dots,\mathbf{e}_d)$

(1.3)
$$\mathbf{L} = \{\mathbf{l} = \mathbf{e}_1 l^1 + \dots + \mathbf{e}_d l^d\ :\ (l^1,\dots,l^d)\in\mathbb{Z}^d\},$$

where $\mathbb{Z}$ denotes the set of integers. The linear transform $\mathbf{A}$ of integer entries maps the lattice $\mathbf{L}$ into itself, leaving the scalar product, or the symmetric Gramian $(\mathbf{g}_{ij})$ of $\mathbf{L}$ invariant (i.e. $\mathbf{A}$ is orthogonal).

2 D-V cell and fundamental domain

The *fundamental domain* $\mathcal{F}$ *of space group* Γ can be defined as follows:

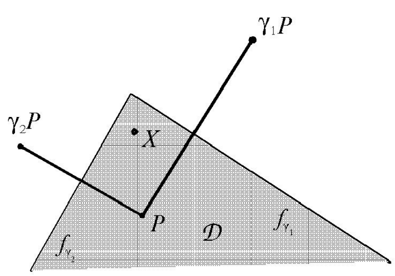


Figure 2.1

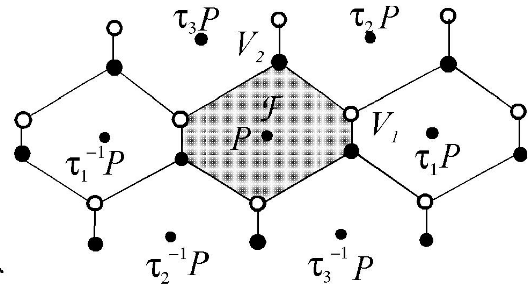


Figure 2.2

$$\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F} = \mathbb{E}^d \quad \text{and} \quad \text{Int}\mathcal{F} \cap \text{Int}\gamma\mathcal{F} = \emptyset \quad \text{for any} \quad \gamma \in \Gamma \setminus \{\mathbf{1}\} \tag{2.1}$$

(“Int” abreviates interior, $\gamma\mathcal{F}$ is the γ-image of $\mathcal{F}$, $\mathbf{1}$ denotes the identity map).
The D-V *cell of the kernel point* P to its Γ-orbit is

$$\mathcal{D}_\Gamma(P) = \{X \in \mathbb{E}^d \, : \, \rho(P,X) \le \rho(\gamma P, X) \quad \text{for each} \quad \gamma \in \Gamma\} \tag{2.2}$$

(Fig. 2.1 Fig. 2.2). We know that $\mathcal{D}$ is intersection of finitely many closed half-spaces. Each of them is bounded by a $(d-1)$-plane (hyperplane).

If $Stab_\Gamma(P) = \mathbf{1}$, i.e. the *stabilizer subgroup* in Γ, fixing the point P, is trivial, then $\mathcal{D}(P) = \mathcal{F}$ is a fundamental domain for Γ. Then any (bisector) hyperface (facet) f_γ to P and γP has a pair $f_{\gamma^{-1}}$ to P and $\gamma^{-1}P$, so that a generator pair of Γ can be obtained as follows

$$\gamma \, : \, f_{\gamma^{-1}} \mapsto f_\gamma, \qquad \gamma^{-1} \, : \, f_\gamma \mapsto f_{\gamma^{-1}}. \tag{2.3}$$

We allow involutive generator $\delta = \delta^{-1}$ ($\delta^2 = \mathbf{1}$, $\delta \neq \mathbf{1}$) as well, when the facet $f_\delta = f_{\delta^{-1}}$ of $\mathcal{F}$ is paired with itself. Only $\mathcal{F}$, representing $\mathbf{1}$, with its facet pairings, denoted by $\mathcal{I}$ (identifications), and the induced $\mathcal{I}$-equivalence of $(d-2)$-faces of $\mathcal{F}$ characterizes the space group Γ as we illustrate this by Fig. 2.2.

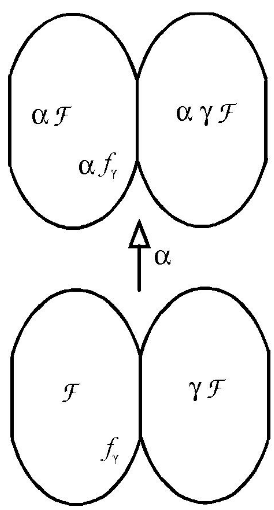


Figure 2.3

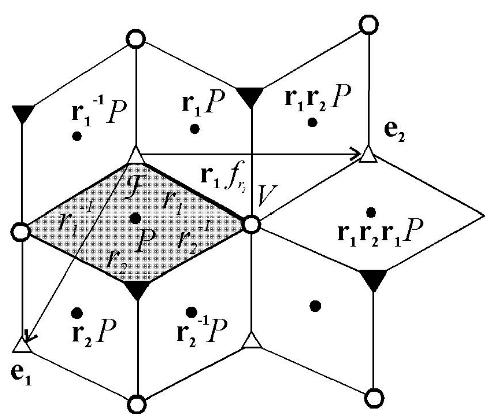


Figure 2.4

Any relation of Γ, in general, can be described by walking from $\mathcal{F}$ through its image domains by crossing side facets and returning to $\mathcal{F}$. Fig. 2.3 shows the main observation to this. If we are in the image domain $\alpha\mathcal{F}$ and cross its side facet αf_γ, which is the α-image of the facet f_γ of $\mathcal{F}$ corresponding to the generator γ, then we arrive at the image domain $\alpha\gamma\mathcal{F}$.

The above α is a product of $\mathcal{F}$-generators from $\mathcal{I}$ (and their inverses), of course. We get so long relation as many facets we have crossed when we walk round from $\mathcal{F}$ into itself.

The special Poincaré algorithm by passing round the $(d-2)$-faces of $\mathcal{F}$, to get defining relations for the group Γ, is a very effective method in the geometric (combinatorial) group theory [M92b].

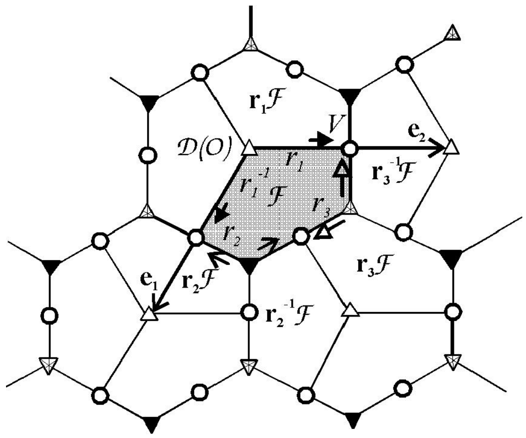


Figure 2.5

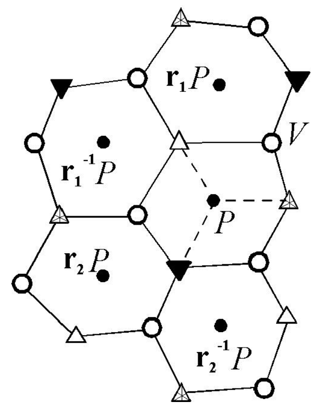


Figure 2.6

We illustrate this situation in Fig. 2.4, where the $\mathbb{E}^2$-group $\Gamma = \mathbf{p3}$ (no.13 in citeit) . The fundamental domain $\mathcal{F} = \mathcal{D}(P)$ is a rhombus by two pairs of sides $f_{r_1^{-1}}$, f_{r_1} and $f_{r_2^{-1}}$, f_{r_2}. We can write

(2.4)
$$P\begin{pmatrix} 1/3 \\ 1/6 \\ 1 \end{pmatrix}, \quad \mathbf{r}_1 : \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{r}_2 : \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$f_{r_1^{-1}} \mapsto f_{r_1}, \quad f_{r_2^{-1}} \mapsto f_{r_2}$$

with respect to the coordinate system $(O; \mathbf{e}_1; \mathbf{e}_2)$ with Gramian

(2.5)
$$(g_{ij}) = (\langle \mathbf{e}_i, \mathbf{e}_j \rangle) = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix};$$

The vertex class $\circ$ of V, provides six images of $\mathcal{F}$ round V and a relation $\mathbf{1} = \mathbf{r}_1\mathbf{r}_2\mathbf{r}_1\mathbf{r}_2\mathbf{r}_1\mathbf{r}_2$, according to the fact that V is also a 3-fold rotation centre to $\mathbf{p3}$.
The presentation of Γ by $\mathcal{F}$: $\mathbf{p3} = \left(\mathbf{r}_1, \mathbf{r}_2 — \mathbf{r}_1^3, \mathbf{r}_2^3, (\mathbf{r}_1\mathbf{r}_2)^3\right)$.

If $Stab_\Gamma(P)$ is of order p, then the D-V cell

(2.6)
$$\mathcal{D}_\Gamma(P) = \bigcup_{\gamma \in Stab(P)} \gamma\mathcal{F}$$

is a union of p images of a fundamental domain $\mathcal{F}$. Namely we take a point Q not fixed at any element of $Stab(P)$. Then the D-V cell of Q under $Stab(P)$

(2.7)
$$\mathcal{D}_{Stab(P)}(Q) := \left\{Y : \rho(Q, Y) \leq \rho(\sigma(Q), Y), \forall\sigma \in Stab(P)\right\}$$
$$\text{provides} \quad \mathcal{F}_\Gamma := \mathcal{D}_\Gamma(P) \cap \mathcal{D}_{Stab(P)}(Q)$$

as a fundamental domain for Γ. $\mathcal{D}_{Stab(P)}(Q)$ is a pyramidal domain with apex P which will be intersected by $\mathcal{D}_\Gamma(P)$. Our example is the plane group $\Gamma = \mathbf{p3}$, again in Fig. 2.5. Here the D-V cell of O is a regular hexagon $\mathcal{D}(O)$.$Stab(O)$ is the group of 3-fold rotations. Choosing a point $Q\begin{pmatrix} q \\ q \\ 1 \end{pmatrix}$, $q > 0$, the pyramidal domain $\mathcal{D}_{Stab(P)}(Q)$ is an angular sector which intersects $\mathcal{D}(O)$ in a pentagonal fundamental domain $\mathcal{F}$ in Fig. 2.5. The arrows $-\!\blacktriangleright$, $\rightarrow$, $-\!\triangleright$ show the pairings of $\mathcal{F}$ respectively, where a geometric side of $\mathcal{F}$ falls into two (algebraic) parts.

3 Fundamental domain for space group XIX.27/01/01

The Gramian matrix in the decagonal family is:

$$G_d = \begin{pmatrix} a & b & -\frac{1}{2}(a+2b) & -\frac{1}{2}(a+2b) \\ b & a & b & -\frac{1}{2}(a+2b) \\ -\frac{1}{2}(a+2b) & b & a & b \\ -\frac{1}{2}(a+2b) & -\frac{1}{2}(a+2b) & b & a \end{pmatrix}. \tag{3.1}$$

To simplify our discussion, $b = -\frac{1}{4}a$ will be assumed, since we want to find a fundamental domain as simple as possible. The point group Γ_{05} of space group $27/01/01 = \Gamma_5$ has one generator: $\gamma_\mathbf{5}$, mapping $\mathbf{e_1} \to \mathbf{e_4} \to \mathbf{e_2} \to (-\mathbf{e_1} - \mathbf{e_2} - \mathbf{e_3} - \mathbf{e_4}) \to \mathbf{e_3} \to \mathbf{e_1}$ cyclically, thus $\gamma_\mathbf{5}$ is a transform of order five, indeed.

3.1 Algorithm for the fundamental domain ($\mathcal{F}_5$) of space group Γ_5:

First: We choose the kernel point $P_1\,(1;1;1;1)$, and determine the *3-plane bisectors of* P_1 *and* γP_1, $\gamma \in \Gamma_{05}$ (P_1 lies in negative halfspaces of the former *3-planes*).
The transforms and the equations of 3-planes will be

$$\gamma_\mathbf{5}: \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -z = 0; \quad \gamma_\mathbf{5}^\mathbf{2}: \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -x = 0;$$

$$\gamma_\mathbf{5}^\mathbf{3}: \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -w = 0; \quad \gamma_\mathbf{5}^\mathbf{4}: \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -y = 0.$$

Thus we get a pyramidal domain $\mathcal{F}_{05}$ with apex O that will contain $\mathcal{F}_5$.

Second: We determine the equations of *bisector 3-planes of the origin* O *and its* Γ_5*-images, now the lattice points with* $0, 1, -1$ *coordinates* . In case of Γ_5 it will be enough to take points with $0, 1$ coordinates. The origin O always lies in negative halfspaces of the above 3-planes. Thus we can intersect the pyramid by new 3-planes
The lattice points and the equations of 3-planes:

$(1;1;1;1;1)^T$, $[1;1;1;1;-2]\cdot(x;y;z;w;1)^T \equiv x+y+z+w-2=0$ with the columns written in transposed form denoted by T up; and we similarly get

$$\begin{aligned}
&(1;0;0;0;1)^T, 4x-y-z-w-2=0,\\
&(0;1;0;0;1)^T, -x+4y-z-w-2=0,\\
&(0;0;1;0;1)^T, -x-y+4z-w-2=0,\\
&(0;0;0;1;1)^T - x-y-z+4w-2=0,\\
&(1;1;0;0;1)^T, 3x+3y-2z-2w-3=0,\\
&(1;0;1;0;1)^T, 3x-2y+3z-2w-3=0,\\
&(1;0;0;1;1)^T, 3x-2y-2z+3w-3=0,\\
&(0;1;1;0;1)^T, -2x+3y+3z-2w-3=0,\\
&(0;1;0;1;1)^T, -2x+3y-2z+3w-3=0,\\
&(0;0;1;1;1)^T, -2x-2y+3z+3w-3=0,\\
&(1;1;1;0;1)^T, 2x+2y+2z-3w-3=0,\\
&(1;1;0;1;1)^T, 2x+2y-3z+2w-3=0,\\
&(1;0;1;1;1)^T, 2x-3y+2z+2w-3=0,\\
&(0;1;1;1;1)^T, -3x+2y+2z+2w-3=0.
\end{aligned} \tag{3.2}$$

Third: We start with five suitable equations of 3-planes and determine the *5 vertices* of a starting simplex (*5-cell*). We take a new 3-plane, substitute the coordinates of all vertices of the convex *4-dimensional polyhedron* into its equation. If at least one vertex exists in the pozitive halfspace of the 3-plane, then we cut the polyhedron, otherwise leave it.

Finally, the fundamental domain $\mathcal{F}_5$ *has* **19** *geometric 3-faces and* **65** *proper vertices.*

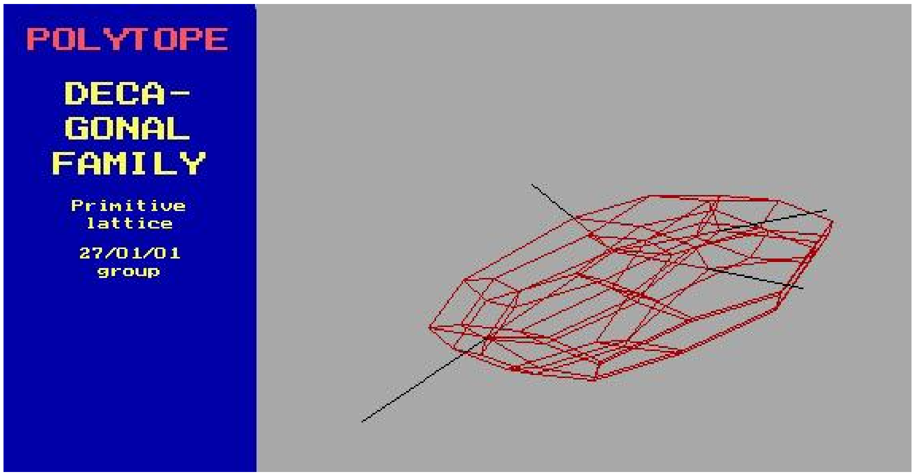


Figure 3.1

3.2 Pairings on fundamental domain $\mathcal{F}_5$

- In the case of 3-faces obtained with the elements of point group Γ_{05}, the pairing transforms are indicated in **Table 1** with usual point coordinates.

Table 3.1

3-planes and points paired with the elements of point group Γ_{05}

Source 3-plane and its points	*Pairing transform*	*Image 3-plane and its points*
$-y=0$ $\left(\frac{7}{10};0;\frac{1}{2};\frac{3}{10}\right),\left(\frac{7}{10};0;\frac{3}{10};\frac{1}{2}\right)$ $\left(\frac{1}{2};0;\frac{7}{10};\frac{3}{10}\right),\left(\frac{1}{2};0;\frac{3}{10};\frac{7}{10}\right)$ $\left(\frac{3}{10};0;\frac{7}{10};\frac{1}{2}\right),\left(\frac{3}{10};0;\frac{1}{2};\frac{7}{10}\right)$ $\left(\frac{3}{5};0;\frac{2}{5};0\right),\left(\frac{3}{5};0;0;\frac{2}{5}\right)$ $\left(\frac{2}{5};0;\frac{3}{5};0\right),\left(\frac{2}{5};0;0;\frac{3}{5}\right)$ $\left(0;0;\frac{3}{5};\frac{2}{5}\right),\left(0;0;\frac{2}{5};\frac{3}{5}\right)$ $\left(\frac{1}{2};0;0;0\right),\left(0;0;\frac{1}{2};0\right)$ $\left(0;0;0;\frac{1}{2}\right),\left(0;0;0;0\right)$	$\gamma_{\mathbf{5}}=\gamma_{\mathbf{5}}^{\mathbf{-4}}$ $\begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-z=0$ $\left(\frac{1}{2};\frac{3}{10};0;\frac{7}{10}\right),\left(\frac{3}{10};\frac{1}{2};0;\frac{7}{10}\right)$ $\left(\frac{7}{10};\frac{3}{10};0;\frac{1}{2}\right),\left(\frac{3}{10};\frac{7}{10};0;\frac{1}{2}\right)$ $\left(\frac{7}{10};\frac{1}{2};0;\frac{3}{10}\right),\left(\frac{1}{2};\frac{7}{10};0;\frac{3}{10}\right)$ $\left(\frac{2}{5};0;0;\frac{3}{5}\right),\left(0;\frac{2}{5};0;\frac{3}{5}\right)$ $\left(\frac{3}{5};0;0;\frac{2}{5}\right),\left(0;\frac{3}{5};0;\frac{2}{5}\right)$ $\left(\frac{3}{5};\frac{2}{5};0;0\right),\left(\frac{2}{5};\frac{3}{5};0;0\right)$ $\left(0;0;0;\frac{1}{2}\right),\left(\frac{1}{2};0;0;0\right)$ $\left(0;\frac{1}{2};0;0\right),\left(0;0;0;0\right)$
$-w=0$ $\left(\frac{7}{10};\frac{1}{2};\frac{3}{10};0\right),\left(\frac{7}{10};\frac{3}{10};\frac{1}{2};0\right)$ $\left(\frac{1}{2};\frac{7}{10};\frac{3}{10};0\right),\left(\frac{1}{2};\frac{3}{10};\frac{7}{10};0\right)$ $\left(\frac{3}{10};\frac{7}{10};\frac{1}{2};0\right),\left(\frac{3}{10};\frac{1}{2};\frac{7}{10};0\right)$ $\left(\frac{3}{5};\frac{2}{5};0;0\right),\left(\frac{3}{5};0;\frac{2}{5};0\right)$ $\left(\frac{2}{5};\frac{3}{5};0;0\right),\left(\frac{2}{5};0;\frac{3}{5};0\right)$ $\left(0;\frac{3}{5};\frac{2}{5};0\right),\left(0;\frac{2}{5};\frac{3}{5};0\right)$ $\left(\frac{1}{2};0;0;0\right),\left(0;\frac{1}{2};0;0\right)$ $\left(0;0;\frac{1}{2};0\right),\left(0;0;0;0\right)$	$\gamma_{\mathbf{5}}^{\mathbf{2}}=\gamma_{\mathbf{5}}^{\mathbf{-3}}$ $\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-x=0$ $\left(0;\frac{7}{10};\frac{1}{2};\frac{3}{10}\right),\left(0;\frac{7}{10};\frac{3}{10};\frac{1}{2}\right)$ $\left(0;\frac{1}{2};\frac{7}{10};\frac{3}{10}\right),\left(0;\frac{1}{2};\frac{3}{10};\frac{7}{10}\right)$ $\left(0;\frac{3}{10};\frac{7}{10};\frac{1}{2}\right),\left(0;\frac{3}{10};\frac{1}{2};\frac{7}{10}\right)$ $\left(0,\frac{3}{5};\frac{2}{5};0\right),\left(0;\frac{3}{5};0;\frac{2}{5}\right)$ $\left(0;\frac{2}{5};\frac{3}{5};0\right),\left(0;\frac{2}{5};0;\frac{3}{5}\right)$ $\left(0;0;\frac{3}{5};\frac{2}{5}\right),\left(0;0;\frac{2}{5};\frac{3}{5}\right)$ $\left(0;\frac{1}{2};0;0\right),\left(0;0;\frac{1}{2};0\right)$ $\left(0;0;0;\frac{1}{2}\right),\left(0;0;0;0\right)$

- Pairing of lattice point bisector 3-planes: Let s_1 be any *bisector 3-face* of the origin and a lattice point Q_1, here $s_1 \subset \mathcal{F}_5$, and $Q_1^{'}$ is the central inversion image of Q_1. We find a transform $\gamma_1 \in \Gamma_{05}$, so that $Q_2 = \gamma_1 Q_1^{'}$, and let $s_2 \subset \mathcal{F}_5$ be the *bisector 3-face* of the origin and the lattice point Q_2. The pairing transforms between the two 3-faces will be the products $\gamma_{12} = \gamma_1 \cdot \begin{pmatrix} \mathbf{1} & -\mathbf{t}_1 \\ 0 & 1 \end{pmatrix}$ and $\gamma_{21} = \gamma_2 \cdot \begin{pmatrix} \mathbf{1} & -\mathbf{t}_2 \\ 0 & 1 \end{pmatrix} = \gamma_{12}^{-1}$, respectively, where $\mathbf{1}$ is the 4x4 identity matrix and $\mathbf{t}_1 = \overrightarrow{OQ_1}$, $\mathbf{t}_2 = \overrightarrow{OQ_2}$, $\gamma_2 \in \Gamma_{05}$. After computing the images of vertices with pairing transforms, we keep those point pairs, that lie in $\mathcal{F}_5$.

Remark 1.: This procedure is an algebraic pairing on $\mathcal{F}_5$, because one geometric 3-face often needs separate parts of more 3-subfaces, and all have different pairs.

Remark 2.: The general scheme for pairing 3-facets s_1 and s_2 is as follows:

$\mathcal{F}_5 \supset s_1$ to $Q_1 = q_1 O \Rightarrow Q_1' = q_1^{-1} O \mapsto \gamma_1 q_1^{-1} O = Q_2$ to $s_2 \subset \mathcal{F}_5$ with $q_1 \in \Gamma_5, \gamma_1 \in \Gamma_{05}$ lead to $\gamma_{12} = \gamma_1 q_1^{-1} : Q_1 \mapsto O \mapsto Q_2, s_1 \mapsto s_2$; and

$\mathcal{F}_5 \supset s_2$ to $Q_2 = q_2 O \Rightarrow Q_2' = q_2^{-1} O \mapsto \gamma_2 q_2^{-1} O = Q_1$ to $s_1 \subset \mathcal{F}_5$ with $q_2 \in \Gamma_5, \gamma_2 \in \Gamma_{05}$ lead to $\gamma_{21} = \gamma_2 q_2^{-1} : Q_2 \mapsto O \mapsto Q_1, s_2 \mapsto s_1$.

Hence $\gamma_{21} = \gamma_{12}^{-1}$ and $\gamma_2 q_2^{-1} = q_1 \gamma_1^{-1}$ hold.

Table 3.2

Algebraic pairing of lattice points bisector 3-planes and their points

Source 3-plane and its points	*Pairing transform*	*Image 3-plane and its points*
$4x-y-z-w-2=0$ $(\frac{4}{5};\frac{3}{5};\frac{2}{5};\frac{1}{5}), (\frac{4}{5};\frac{3}{5};\frac{1}{5};\frac{2}{5})$ $(\frac{4}{5};\frac{2}{5};\frac{3}{5};\frac{1}{5}), (\frac{4}{5};\frac{2}{5};\frac{1}{5};\frac{3}{5})$ $(\frac{4}{5};\frac{1}{5};\frac{3}{5};\frac{2}{5}), (\frac{4}{5};\frac{1}{5};\frac{2}{5};\frac{3}{5})$ $(\frac{7}{10};\frac{1}{2};\frac{3}{10};0), (\frac{7}{10};\frac{1}{2};0;\frac{3}{10})$ $(\frac{7}{10};\frac{3}{10};\frac{1}{2};0), (\frac{7}{10};\frac{3}{10};0;\frac{1}{2})$ $(\frac{7}{10};0;\frac{1}{2};\frac{3}{10}), (\frac{7}{10};0;\frac{3}{10};\frac{1}{2})$ $(\frac{3}{5};\frac{2}{5};0;0), (\frac{3}{5};0;\frac{2}{5};0)$ $(\frac{3}{5};0;0;\frac{2}{5}), (\frac{1}{2};0;0;0)$	$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$x+y+z+w-2=0$ $(\frac{4}{5};\frac{3}{5};\frac{2}{5};\frac{1}{5}), (\frac{4}{5};\frac{2}{5};\frac{3}{5};\frac{1}{5})$ $(\frac{3}{5};\frac{4}{5};\frac{2}{5};\frac{1}{5}), (\frac{3}{5};\frac{2}{5};\frac{4}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{4}{5};\frac{3}{5};\frac{1}{5}), (\frac{2}{5};\frac{3}{5};\frac{4}{5};\frac{1}{5})$ $(\frac{4}{5};\frac{3}{5};\frac{3}{10};\frac{3}{10}), (\frac{4}{5};\frac{3}{10};\frac{3}{5};\frac{3}{10})$ $(\frac{3}{5};\frac{4}{5};\frac{3}{10};\frac{3}{10}), (\frac{3}{5};\frac{3}{10};\frac{4}{5};\frac{3}{10})$ $(\frac{3}{10};\frac{4}{5};\frac{3}{5};\frac{3}{10}), (\frac{3}{10};\frac{3}{5};\frac{4}{5};\frac{3}{10})$ $(\frac{4}{5};\frac{2}{5};\frac{2}{5};\frac{2}{5}), (\frac{2}{5};\frac{4}{5};\frac{2}{5};\frac{2}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{4}{5};\frac{2}{5}), (\frac{1}{2};\frac{1}{2};\frac{1}{2};\frac{1}{2})$
$-x+4y-z-w-2=0$ $(\frac{1}{5};\frac{4}{5};\frac{3}{5};\frac{2}{5}), (\frac{2}{5};\frac{4}{5};\frac{3}{5};\frac{1}{5})$ $(\frac{1}{5};\frac{4}{5};\frac{2}{5};\frac{3}{5}), (\frac{3}{5};\frac{4}{5};\frac{2}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{4}{5};\frac{1}{5};\frac{3}{5}), (\frac{3}{5};\frac{4}{5};\frac{1}{5};\frac{2}{5})$ $(0;\frac{7}{10};\frac{1}{2};\frac{3}{10}), (\frac{3}{10};\frac{7}{10};\frac{1}{2};0)$ $(0;\frac{7}{10};\frac{3}{10};\frac{1}{2}), (\frac{1}{2};\frac{7}{10};\frac{3}{10};0)$ $(\frac{3}{10};\frac{7}{10};0;\frac{1}{2}), (\frac{1}{2};\frac{7}{10};0;\frac{3}{10})$ $(0;\frac{3}{5};\frac{2}{5};0), (0;\frac{3}{5};0;\frac{2}{5})$ $(\frac{2}{5};\frac{3}{5};0;0), (0;\frac{1}{2};0;0)$	$\begin{pmatrix} 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$x+y+z+w-2=0$ $(\frac{4}{5};\frac{3}{5};\frac{1}{5};\frac{2}{5}), (\frac{4}{5};\frac{2}{5};\frac{1}{5};\frac{3}{5})$ $(\frac{3}{5};\frac{4}{5};\frac{1}{5};\frac{2}{5}), (\frac{3}{5};\frac{2}{5};\frac{1}{5};\frac{4}{5})$ $(\frac{2}{5};\frac{4}{5};\frac{1}{5};\frac{3}{5}), (\frac{2}{5};\frac{3}{5};\frac{1}{5};\frac{4}{5})$ $(\frac{4}{5};\frac{3}{5};\frac{3}{10};\frac{3}{10}), (\frac{4}{5};\frac{3}{10};\frac{3}{10};\frac{3}{5})$ $(\frac{3}{5};\frac{4}{5};\frac{3}{10};\frac{3}{10}), (\frac{3}{5};\frac{3}{10};\frac{3}{10};\frac{4}{5})$ $(\frac{3}{10};\frac{4}{5};\frac{3}{10};\frac{3}{5}), (\frac{3}{10};\frac{3}{5};\frac{3}{10};\frac{4}{5})$ $(\frac{4}{5};\frac{2}{5};\frac{2}{5};\frac{2}{5}), (\frac{2}{5};\frac{4}{5};\frac{2}{5};\frac{2}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{4}{5}), (\frac{1}{2};\frac{1}{2};\frac{1}{2};\frac{1}{2})$
$-x-y+4z-w-2=0$ $(\frac{2}{5};\frac{1}{5};\frac{4}{5};\frac{3}{5}), (\frac{1}{5};\frac{2}{5};\frac{4}{5};\frac{3}{5})$ $(\frac{3}{5};\frac{1}{5};\frac{4}{5};\frac{2}{5}), (\frac{1}{5};\frac{3}{5};\frac{4}{5};\frac{2}{5})$ $(\frac{3}{5};\frac{2}{5};\frac{4}{5};\frac{1}{5}), (\frac{2}{5};\frac{3}{5};\frac{4}{5};\frac{1}{5})$ $(\frac{3}{10};0;\frac{7}{10};\frac{1}{2}), (0;\frac{3}{10};\frac{7}{10};\frac{1}{2})$ $(\frac{1}{2};0;\frac{7}{10};\frac{3}{10}), (0;\frac{1}{2};\frac{7}{10};\frac{3}{10})$ $(\frac{1}{2};\frac{3}{10};\frac{7}{10};0), (\frac{3}{10};\frac{1}{2};\frac{7}{10};0)$ $(0;0;\frac{3}{5};\frac{2}{5}), (\frac{2}{5};0;\frac{3}{5};0)$ $(0;\frac{2}{5};\frac{3}{5};0), (0;0;\frac{1}{2};0)$	$\begin{pmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$x+y+z+w-2=0$ $(\frac{4}{5};\frac{1}{5};\frac{3}{5};\frac{2}{5}), (\frac{4}{5};\frac{1}{5};\frac{2}{5};\frac{3}{5})$ $(\frac{3}{5};\frac{1}{5};\frac{4}{5};\frac{2}{5}), (\frac{3}{5};\frac{1}{5};\frac{2}{5};\frac{4}{5})$ $(\frac{2}{5};\frac{1}{5};\frac{4}{5};\frac{3}{5}), (\frac{2}{5};\frac{1}{5};\frac{3}{5};\frac{4}{5})$ $(\frac{4}{5};\frac{3}{10};\frac{3}{5};\frac{3}{10}), (\frac{4}{5};\frac{3}{10};\frac{3}{10};\frac{3}{5})$ $(\frac{3}{5};\frac{3}{10};\frac{4}{5};\frac{3}{10}), (\frac{3}{5};\frac{3}{10};\frac{3}{10};\frac{4}{5})$ $(\frac{3}{10};\frac{3}{10};\frac{4}{5};\frac{3}{5}), (\frac{3}{10};\frac{3}{10};\frac{3}{5};\frac{4}{5})$ $(\frac{4}{5};\frac{2}{5};\frac{2}{5};\frac{2}{5}), (\frac{2}{5};\frac{2}{5};\frac{4}{5};\frac{2}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{4}{5}), (\frac{1}{2};\frac{1}{2};\frac{1}{2};\frac{1}{2})$
$-x-y-z+4w-2=0$ $(\frac{3}{5};\frac{2}{5};\frac{1}{5};\frac{4}{5}), (\frac{3}{5};\frac{1}{5};\frac{2}{5};\frac{4}{5})$ $(\frac{2}{5};\frac{3}{5};\frac{1}{5};\frac{4}{5}), (\frac{2}{5};\frac{1}{5};\frac{3}{5};\frac{4}{5})$ $(\frac{1}{5};\frac{3}{5};\frac{2}{5};\frac{4}{5}), (\frac{1}{5};\frac{2}{5};\frac{3}{5};\frac{4}{5})$ $(\frac{1}{2};\frac{3}{10};0;\frac{7}{10}), (\frac{1}{2};0;\frac{3}{10};\frac{7}{10})$ $(\frac{3}{10};\frac{1}{2};0;\frac{7}{10}), (\frac{3}{10};0;\frac{1}{2};\frac{7}{10})$ $(0;\frac{1}{2};\frac{3}{10};\frac{7}{10}), (0;\frac{3}{10};\frac{1}{2};\frac{7}{10})$ $(\frac{2}{5};0;0;\frac{3}{5}), (0;\frac{2}{5};0;\frac{3}{5})$ $(0;0;\frac{2}{5};\frac{3}{5}), (0;0;0;\frac{1}{2})$	$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$x+y+z+w-2=0$ $(\frac{1}{5};\frac{4}{5};\frac{3}{5};\frac{2}{5}), (\frac{1}{5};\frac{4}{5};\frac{2}{5};\frac{3}{5})$ $(\frac{1}{5};\frac{3}{5};\frac{4}{5};\frac{2}{5})(\frac{1}{5};\frac{3}{5};\frac{2}{5};\frac{4}{5})$ $(\frac{1}{5};\frac{2}{5};\frac{4}{5};\frac{3}{5}), (\frac{1}{5};\frac{2}{5};\frac{3}{5};\frac{4}{5})$ $(\frac{3}{10};\frac{4}{5};\frac{3}{5};\frac{3}{10}), (\frac{3}{10};\frac{4}{5};\frac{3}{10};\frac{3}{5})$ $(\frac{3}{10};\frac{3}{5};\frac{4}{5};\frac{3}{10}), (\frac{3}{10};\frac{3}{5};\frac{3}{10};\frac{4}{5})$ $(\frac{3}{10};\frac{3}{10};\frac{4}{5};\frac{3}{5}), (\frac{3}{10};\frac{3}{10};\frac{3}{5};\frac{4}{5})$ $(\frac{2}{5};\frac{4}{5};\frac{2}{5};\frac{2}{5}), (\frac{2}{5};\frac{2}{5};\frac{4}{5};\frac{2}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{4}{5}), (\frac{1}{2};\frac{1}{2};\frac{1}{2};\frac{1}{2})$

Table 3.2 (cont.1)

Source 3-plane and its points	*Pairing transform*	*Image 3-plane and its points*
$3x+3y-2z-2w-3=0$ $\left(\tfrac{7}{10};\tfrac{7}{10};\tfrac{2}{5};\tfrac{1}{5}\right),\left(\tfrac{7}{10};\tfrac{7}{10};\tfrac{1}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{4}{5};\tfrac{3}{5};\tfrac{2}{5};\tfrac{1}{5}\right),\left(\tfrac{4}{5};\tfrac{3}{5};\tfrac{1}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{7}{10};\tfrac{1}{2};\tfrac{3}{10};0\right),\left(\tfrac{7}{10};\tfrac{1}{2};0;\tfrac{3}{10}\right)$ $\left(\tfrac{3}{5};\tfrac{2}{5};0;0\right)$	$\begin{pmatrix} 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+2y+2z-3w-3=0$ $\left(\tfrac{7}{10};\tfrac{1}{2};\tfrac{3}{10};0\right),\left(\tfrac{1}{2};\tfrac{7}{10};\tfrac{3}{10};0\right)$ $\left(\tfrac{4}{5};\tfrac{3}{5};\tfrac{2}{5};\tfrac{1}{5}\right),\left(\tfrac{3}{5};\tfrac{4}{5};\tfrac{2}{5};\tfrac{1}{5}\right)$ $\left(\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{2};\tfrac{1}{5}\right),\left(\tfrac{1}{2};\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{5}\right)$ $\left(\tfrac{3}{5};\tfrac{3}{5};\tfrac{3}{5};\tfrac{1}{5}\right)$
$3x+3y-2z-2w-3=0$ $\left(\tfrac{7}{10};\tfrac{7}{10};\tfrac{2}{5};\tfrac{1}{5}\right),\left(\tfrac{7}{10};\tfrac{7}{10};\tfrac{1}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{3}{5};\tfrac{4}{5};\tfrac{2}{5};\tfrac{1}{5}\right),\left(\tfrac{3}{5};\tfrac{4}{5};\tfrac{1}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{1}{2};\tfrac{7}{10};\tfrac{3}{10};0\right),\left(\tfrac{1}{2};\tfrac{7}{10};0;\tfrac{3}{10}\right)$ $\left(\tfrac{2}{5};\tfrac{3}{5};0;0\right)$	$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-3x+2y+2z+2w-3=0$ $\left(0;\tfrac{7}{10};\tfrac{1}{2};\tfrac{3}{10}\right),\left(0;\tfrac{1}{2};\tfrac{7}{10};\tfrac{3}{10}\right)$ $\left(\tfrac{1}{5};\tfrac{4}{5};\tfrac{3}{5};\tfrac{2}{5}\right),\left(\tfrac{1}{5};\tfrac{3}{5};\tfrac{4}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{1}{5};\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{2}\right),\left(\tfrac{1}{5};\tfrac{1}{2};\tfrac{4}{5};\tfrac{1}{2}\right)$ $\left(\tfrac{1}{5};\tfrac{3}{5};\tfrac{3}{5};\tfrac{3}{5}\right)$
$-2x+3y+3z-2w-3=0$ $\left(\tfrac{2}{5};\tfrac{7}{10};\tfrac{7}{10};\tfrac{1}{5}\right),\left(\tfrac{1}{5};\tfrac{7}{10};\tfrac{7}{10};\tfrac{2}{5}\right)$ $\left(\tfrac{2}{5};\tfrac{4}{5};\tfrac{3}{5};\tfrac{1}{5}\right),\left(\tfrac{1}{5};\tfrac{4}{5};\tfrac{3}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{3}{10};\tfrac{7}{10};\tfrac{1}{2};0\right),\left(0;\tfrac{7}{10};\tfrac{1}{2};\tfrac{3}{10}\right)$ $\left(0;\tfrac{3}{5};\tfrac{2}{5};0\right)$	$\begin{pmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+2y+2z-3w-3=0$ $\left(\tfrac{1}{2};\tfrac{3}{10};\tfrac{7}{10};0\right),\left(\tfrac{7}{10};\tfrac{3}{10};\tfrac{1}{2};0\right)$ $\left(\tfrac{3}{5};\tfrac{2}{5};\tfrac{4}{5};\tfrac{1}{5}\right),\left(\tfrac{4}{5};\tfrac{2}{5};\tfrac{3}{5};\tfrac{1}{5}\right)$ $\left(\tfrac{1}{2};\tfrac{1}{2};\tfrac{4}{5};\tfrac{1}{5}\right),\left(\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{2};\tfrac{1}{5}\right)$ $\left(\tfrac{3}{5};\tfrac{3}{5};\tfrac{3}{5};\tfrac{1}{5}\right)$
$-2x+3y+3z-2w-3=0$ $\left(\tfrac{2}{5};\tfrac{7}{10};\tfrac{7}{10};\tfrac{1}{5}\right),\left(\tfrac{1}{5};\tfrac{7}{10};\tfrac{7}{10};\tfrac{2}{5}\right)$ $\left(\tfrac{2}{5};\tfrac{3}{5};\tfrac{4}{5};\tfrac{1}{5}\right),\left(\tfrac{1}{5};\tfrac{3}{5};\tfrac{4}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{3}{10};\tfrac{1}{2};\tfrac{7}{10};0\right),\left(0;\tfrac{1}{2};\tfrac{7}{10};\tfrac{3}{10}\right)$ $\left(0;\tfrac{2}{5};\tfrac{3}{5};0\right)$	$\begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-3x+2y+2z+2w-3=0$ $\left(0;\tfrac{1}{2};\tfrac{3}{10};\tfrac{7}{10}\right),\left(0;\tfrac{7}{10};\tfrac{3}{10};\tfrac{1}{2}\right)$ $\left(\tfrac{1}{5};\tfrac{3}{5};\tfrac{2}{5};\tfrac{4}{5}\right),\left(\tfrac{1}{5};\tfrac{4}{5};\tfrac{2}{5};\tfrac{3}{5}\right)$ $\left(\tfrac{1}{5};\tfrac{1}{2};\tfrac{1}{2};\tfrac{4}{5}\right),\left(\tfrac{1}{5};\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{2}\right)$ $\left(\tfrac{1}{5};\tfrac{3}{5};\tfrac{3}{5};\tfrac{3}{5}\right)$
$-2x-2y+3z+3w-3=0$ $\left(\tfrac{2}{5};\tfrac{1}{5};\tfrac{7}{10};\tfrac{7}{10}\right),\left(\tfrac{1}{5};\tfrac{2}{5};\tfrac{7}{10};\tfrac{7}{10}\right)$ $\left(\tfrac{2}{5};\tfrac{1}{5};\tfrac{4}{5};\tfrac{3}{5}\right),\left(\tfrac{1}{5};\tfrac{2}{5};\tfrac{4}{5};\tfrac{3}{5}\right)$ $\left(\tfrac{3}{10};0;\tfrac{7}{10};\tfrac{1}{2}\right),\left(0;\tfrac{3}{10};\tfrac{7}{10};\tfrac{1}{2}\right)$ $\left(0;0;\tfrac{3}{5};\tfrac{2}{5}\right)$	$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+2y+2z-3w-3=0$ $\left(\tfrac{3}{10};\tfrac{7}{10};\tfrac{1}{2};0\right),\left(\tfrac{3}{10};\tfrac{1}{2};\tfrac{7}{10};0\right)$ $\left(\tfrac{2}{5};\tfrac{4}{5};\tfrac{3}{5};\tfrac{1}{5}\right),\left(\tfrac{2}{5};\tfrac{3}{5};\tfrac{4}{5};\tfrac{1}{5}\right)$ $\left(\tfrac{1}{2};\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{5}\right),\left(\tfrac{1}{2};\tfrac{1}{2};\tfrac{4}{5};\tfrac{1}{5}\right)$ $\left(\tfrac{3}{5};\tfrac{3}{5};\tfrac{3}{5};\tfrac{1}{5}\right)$
$-2x-2y+3z+3w-3=0$ $\left(\tfrac{2}{5};\tfrac{1}{5};\tfrac{7}{10};\tfrac{7}{10}\right),\left(\tfrac{1}{5};\tfrac{2}{5};\tfrac{7}{10};\tfrac{7}{10}\right)$ $\left(\tfrac{2}{5};\tfrac{1}{5};\tfrac{3}{5};\tfrac{4}{5}\right),\left(\tfrac{1}{5};\tfrac{2}{5};\tfrac{3}{5};\tfrac{4}{5}\right)$ $\left(\tfrac{3}{10};0;\tfrac{1}{2};\tfrac{7}{10}\right),\left(0;\tfrac{3}{10};\tfrac{1}{2};\tfrac{7}{10}\right)$ $\left(0;0;\tfrac{2}{5};\tfrac{3}{5}\right)$	$\begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-3x+2y+2z+2w-3=0$ $\left(0;\tfrac{3}{10};\tfrac{7}{10};\tfrac{1}{2}\right),\left(0;\tfrac{3}{10};\tfrac{1}{2};\tfrac{7}{10}\right)$ $\left(\tfrac{1}{5};\tfrac{2}{5};\tfrac{4}{5};\tfrac{3}{5}\right),\left(\tfrac{1}{5};\tfrac{2}{5};\tfrac{3}{5};\tfrac{4}{5}\right)$ $\left(\tfrac{1}{5};\tfrac{1}{2};\tfrac{4}{5};\tfrac{1}{2}\right),\left(\tfrac{1}{5};\tfrac{1}{2};\tfrac{1}{2};\tfrac{4}{5}\right)$ $\left(\tfrac{1}{5};\tfrac{3}{5};\tfrac{3}{5};\tfrac{3}{5}\right)$
$3x-2y+3z-2w-3=0$ $\left(\tfrac{7}{10};\tfrac{2}{5};\tfrac{7}{10};\tfrac{1}{5}\right),\left(\tfrac{7}{10};\tfrac{1}{5};\tfrac{7}{10};\tfrac{2}{5}\right)$ $\left(\tfrac{4}{5};\tfrac{2}{5};\tfrac{3}{5};\tfrac{1}{5}\right),\left(\tfrac{4}{5};\tfrac{1}{5};\tfrac{3}{5};\tfrac{2}{5}\right)$ $\left(\tfrac{7}{10};\tfrac{3}{10};\tfrac{1}{2};0\right),\left(\tfrac{7}{10};0;\tfrac{1}{2};\tfrac{3}{10}\right)$ $\left(\tfrac{3}{5};0;\tfrac{2}{5};0\right)$	$\begin{pmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+2y-3z+2w-3=0$ $\left(\tfrac{1}{2};\tfrac{3}{10};0;\tfrac{7}{10}\right),\left(\tfrac{7}{10};\tfrac{3}{10};0;\tfrac{1}{2}\right)$ $\left(\tfrac{3}{5};\tfrac{2}{5};\tfrac{1}{5};\tfrac{4}{5}\right),\left(\tfrac{4}{5};\tfrac{2}{5};\tfrac{1}{5};\tfrac{3}{5}\right)$ $\left(\tfrac{1}{2};\tfrac{1}{2};\tfrac{1}{5};\tfrac{4}{5}\right),\left(\tfrac{4}{5};\tfrac{1}{2};\tfrac{1}{5};\tfrac{1}{2}\right)$ $\left(\tfrac{3}{5};\tfrac{3}{5};\tfrac{1}{5};\tfrac{3}{5}\right)$

Table 3.2 (cont.2)

Source 3-plane and its points	*Pairing transform*	*Image 3-plane and its points*
$3x-2y+3z-2w-3=0$ $\left(\frac{7}{10};\frac{2}{5};\frac{7}{10};\frac{1}{5}\right),\left(\frac{7}{10};\frac{1}{5};\frac{7}{10};\frac{2}{5}\right)$ $\left(\frac{3}{5};\frac{2}{5};\frac{4}{5};\frac{1}{5}\right),\left(\frac{3}{5};\frac{1}{5};\frac{4}{5};\frac{2}{5}\right)$ $\left(\frac{1}{2};\frac{3}{10};\frac{7}{10};0\right),\left(\frac{1}{2};0;\frac{7}{10};\frac{3}{10}\right)$ $\left(\frac{2}{5};0;\frac{3}{5};0\right)$	$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x-3y+2z+2w-3=0$ $\left(\frac{7}{10};0;\frac{1}{2};\frac{3}{10}\right),\left(\frac{1}{2};0;\frac{7}{10};\frac{3}{10}\right)$ $\left(\frac{4}{5};\frac{1}{5};\frac{3}{5};\frac{2}{5}\right),\left(\frac{3}{5};\frac{1}{5};\frac{4}{5};\frac{2}{5}\right)$ $\left(\frac{4}{5};\frac{1}{5};\frac{1}{2};\frac{1}{2}\right),\left(\frac{1}{2};\frac{1}{5};\frac{4}{5};\frac{1}{2}\right)$ $\left(\frac{3}{5};\frac{1}{5};\frac{3}{5};\frac{3}{5}\right)$
$3x-2y-2z+3w-3=0$ $\left(\frac{7}{10};\frac{2}{5};\frac{1}{5};\frac{7}{10}\right),\left(\frac{7}{10};\frac{1}{5};\frac{2}{5};\frac{7}{10}\right)$ $\left(\frac{3}{5};\frac{2}{5};\frac{1}{5};\frac{4}{5}\right),\left(\frac{3}{5};\frac{1}{5};\frac{2}{5};\frac{4}{5}\right)$ $\left(\frac{1}{2};\frac{3}{10};0;\frac{7}{10}\right),\left(\frac{1}{2};0;\frac{3}{10};\frac{7}{10}\right)$ $\left(\frac{2}{5};0;0;\frac{3}{5}\right)$	$\begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+2y-3z+2w-3=0$ $\left(\frac{7}{10};\frac{1}{2};0;\frac{3}{10}\right),\left(\frac{1}{2};\frac{7}{10};0;\frac{3}{10}\right)$ $\left(\frac{4}{5};\frac{3}{5};\frac{1}{5};\frac{2}{5}\right),\left(\frac{3}{5};\frac{4}{5};\frac{1}{5};\frac{2}{5}\right)$ $\left(\frac{4}{5};\frac{1}{2};\frac{1}{5};\frac{1}{2}\right),\left(\frac{1}{2};\frac{4}{5};\frac{1}{5};\frac{1}{2}\right)$ $\left(\frac{3}{5};\frac{3}{5};\frac{1}{5};\frac{3}{5}\right)$
$3x-2y-2z+3w-3=0$ $\left(\frac{7}{10};\frac{2}{5};\frac{1}{5};\frac{7}{10}\right),\left(\frac{7}{10};\frac{1}{5};\frac{2}{5};\frac{7}{10}\right)$ $\left(\frac{4}{5};\frac{2}{5};\frac{1}{5};\frac{3}{5}\right),\left(\frac{4}{5};\frac{1}{5};\frac{2}{5};\frac{3}{5}\right)$ $\left(\frac{7}{10};\frac{3}{10};0;\frac{1}{2}\right),\left(\frac{7}{10};0;\frac{3}{10};\frac{1}{2}\right)$ $\left(\frac{3}{5};0;0;\frac{2}{5}\right)$	$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x-3y+2z+2w-3=0$ $\left(\frac{3}{10};0;\frac{7}{10};\frac{1}{2}\right),\left(\frac{3}{10};0;\frac{1}{2};\frac{7}{10}\right)$ $\left(\frac{2}{5};\frac{1}{5};\frac{4}{5};\frac{3}{5}\right),\left(\frac{2}{5};\frac{1}{5};\frac{3}{5};\frac{4}{5}\right)$ $\left(\frac{1}{2};\frac{1}{5};\frac{4}{5};\frac{1}{2}\right),\left(\frac{1}{2};\frac{1}{5};\frac{1}{2};\frac{4}{5}\right)$ $\left(\frac{3}{5};\frac{1}{5};\frac{3}{5};\frac{3}{5}\right)$
$-2x+3y-2z+3w-3=0$ $\left(\frac{2}{5};\frac{7}{10};\frac{1}{5};\frac{7}{10}\right),\left(\frac{1}{5};\frac{7}{10};\frac{2}{5};\frac{7}{10}\right)$ $\left(\frac{2}{5};\frac{4}{5};\frac{1}{5};\frac{3}{5}\right),\left(\frac{1}{5};\frac{4}{5};\frac{2}{5};\frac{3}{5}\right)$ $\left(\frac{3}{10};\frac{7}{10};0;\frac{1}{2}\right),\left(0;\frac{7}{10};\frac{3}{10};\frac{1}{2}\right)$ $\left(0;\frac{3}{5};0;\frac{2}{5}\right)$	$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+2y-3z+2w-3=0$ $\left(\frac{3}{10};\frac{7}{10};0;\frac{1}{2}\right),\left(\frac{3}{10};\frac{1}{2};0;\frac{7}{10}\right)$ $\left(\frac{2}{5};\frac{4}{5};\frac{1}{5};\frac{3}{5}\right),\left(\frac{2}{5};\frac{3}{5};\frac{1}{5};\frac{4}{5}\right)$ $\left(\frac{1}{2};\frac{4}{5};\frac{1}{5};\frac{1}{2}\right),\left(\frac{1}{2};\frac{1}{2};\frac{1}{5};\frac{4}{5}\right)$ $\left(\frac{3}{5};\frac{3}{5};\frac{1}{5};\frac{3}{5}\right)$
$-2x+3y-2z+3w-3=0$ $\left(\frac{2}{5};\frac{7}{10};\frac{1}{5};\frac{7}{10}\right),\left(\frac{1}{5};\frac{7}{10};\frac{2}{5};\frac{7}{10}\right)$ $\left(\frac{2}{5};\frac{3}{5};\frac{1}{5};\frac{4}{5}\right),\left(\frac{1}{5};\frac{3}{5};\frac{2}{5};\frac{4}{5}\right)$ $\left(\frac{3}{10};\frac{1}{2};0;\frac{7}{10}\right),\left(0;\frac{1}{2};\frac{3}{10};\frac{7}{10}\right)$ $\left(0;\frac{2}{5};0;\frac{3}{5}\right)$	$\begin{pmatrix} 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x-3y+2z+2w-3=0$ $\left(\frac{1}{2};0;\frac{3}{10};\frac{7}{10}\right),\left(\frac{7}{10};0;\frac{3}{10};\frac{1}{2}\right)$ $\left(\frac{3}{5};\frac{1}{5};\frac{2}{5};\frac{4}{5}\right),\left(\frac{4}{5};\frac{1}{5};\frac{2}{5};\frac{3}{5}\right)$ $\left(\frac{1}{2};\frac{1}{5};\frac{1}{2};\frac{4}{5}\right),\left(\frac{4}{5};\frac{1}{5};\frac{1}{2};\frac{1}{2}\right)$ $\left(\frac{3}{5};\frac{1}{5};\frac{3}{5};\frac{3}{5}\right)$

Now, the relations among these pairing transforms are easy by their algebraic forms. The 2-face classes would provide those much more lengthy, but analogously as to Fig.2.5.

4 Fundamental domain for space group XXII.31/04/02

The icosahedral crystal family has two Bravais lattices with Gramians
$g_{ij} = g_{ji} := (\langle \mathbf{e}_i, \mathbf{e}_j \rangle)$

$$\begin{pmatrix} a & -\frac{a}{4} & -\frac{a}{4} & -\frac{a}{4} \\ -\frac{a}{4} & a & -\frac{a}{4} & -\frac{a}{4} \\ -\frac{a}{4} & -\frac{a}{4} & a & -\frac{a}{4} \\ -\frac{a}{4} & -\frac{a}{4} & -\frac{a}{4} & a \end{pmatrix}, \begin{pmatrix} a & \frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & a & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & a & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & \frac{a}{2} & a \end{pmatrix}, \tag{4.1}$$

inverse to each other up to a positive constant factor a, for the primitive and its SN centred (seitenflächen-nebendiagonal-zentriert) lattices, respectively [BBNWZ]. **XXII.31/04/02** $= \Gamma_4$ is a space group for the SN centred lattice. The point group Γ_{04} of space group Γ_4 has 120 linear transforms, and can be generated by

$$\alpha_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$\alpha_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

of order 4, 5, 6, respectively.

4.1 Algorithm for the fundamental domain $\mathcal{F}_4$ of space group Γ_4

First: We choose a further kernel point $P_1(3;2;1;0)$ to obtain the 3-plane bisectors of these transforms (the kernel always lies in negative halfspaces of the former *3-planes*).
Second: We determine the equations of *bisector 3-planes of a kernel, say the origin O* in the role of P, *and its Γ_4-images, now the lattice points with $0, 1, -1$ coordinates are sufficient for us.*. Then we obtain the 3-faces (facets) of $\mathcal{D}(O)$.
Thus we get a pyramidal domain $\mathcal{F}_{04}$ (with apex O) that will contain $\mathcal{F}_4$.
Third: We apply the algorithm in the third point of the previous chapter, and we obtain the equations of 3-planes and the vertices of the fundamental domain.

Finally, the fundamental domain $\mathcal{F}_4$ will be a famous Coxeter simplex for $\Gamma_4 = 31/04/02$. Among the **5** bisector 3-faces, one bisector is to the origin and the lattice point $(1;0;0;0)$, further four 3-faces appertain to $\mathcal{F}_{04}$. We have chosen this example for simplicity. We can see in [AM02] that a fundamental domain can have a cumbersome structure to be described by computer, in general.

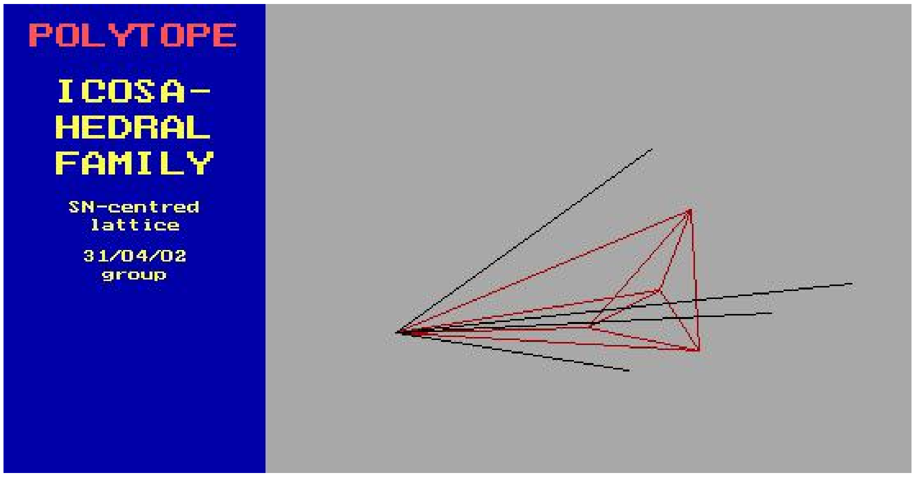


Figure 4.1

4.2 Pairings on fundamental domain $\mathcal{F}_4$

For pairing the bisector 3-planes of $\mathcal{F}_4$, and in general, we consider those images of the kernel which are under transforms of inverse types $(\mathbf{a}, \mathbf{A})$, $(-\mathbf{A}^{-1}\mathbf{a}, \mathbf{A}^{-1})$ in the space group scheme (1.2). This will be applied first for $\mathcal{F}_{Stab(P)}$, then the further bisectors of $\mathcal{F}_4$.
Remark 1.: In case of our fundamental domain $\mathcal{F}_4$ each bisector 3-planes is paired with itself by 3-plane reflection.
Remark 2.: In our case we used two kernel points, but just any P (near O, not lying in the reflection planes) with non integer coordinates would be an appropriate kernel for $\mathcal{F}_4$.

Table 4.1

Pairing on fundamental domain $\mathcal{F}_4$

Source 3-plane and its points	*Pairing transform by 5x5 homegenous matrixes*	*Image 3-plane and its points*
$2x+y+z+w-1=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$	μ_1 $\begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+y+z+w-1=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$
$y-x=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(0;0;0;0)$	μ_2 $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$y-x=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(0;0;0;0)$
$z-y=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$ $(0;0;0;0)$	μ_3 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$z-y=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$ $(0;0;0;0)$
$w-z=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$ $(0;0;0;0)$	μ_4 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$w-z=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$ $(0;0;0;0)$
$-x-y-z-2w=0$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$ $(0;0;0;0)$	μ_5 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-x-y-z-2w=0$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{3}{5};\frac{3}{5};\frac{-2}{5};\frac{-2}{5})$ $(\frac{4}{5};\frac{-1}{5};\frac{-1}{5};\frac{-1}{5})$ $(0;0;0;0)$

Now, we perform an incidence (flag) structure of a D-V cell $\mathcal{D}$ or $\mathcal{F}$ for the space group Γ. A *flag* of a convex polyhedron in $\mathbb{E}^d$ consists of a $(d-1)$-dimensional facet f_1^{d-1}; then an incident $(d-2)$-face f_1^{d-2}, as intersection of two $(d-1)$-facets

(4.3) $$f_1^{d-2} = f_1^{d-1} \cap f_2^{d-1}, \dots,$$

then an incident $(d-k)$-face as intersection

(4.4) $$f_1^{d-k} = f_1^{d-k+1} \cap f_k^{d-1}, \quad 1 \le k \le d.$$

Thus the flag

(4.5) $$F_1 := \left(f_1^0, f_1^1, \dots, f_1^{(d-2)}, f_1^{(d-1)}\right)$$

as a d-tuple of consecutively incident faces.

Flag structure of fundamental domain $\mathcal{F}_4$

$3-faces$

(1) $2x+y+z+w-1=0$ $[1,2,3,4]$

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(2) $y-x=0$ $[1,2,3,5]$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) $z-y=0$ $[1,2,4,5]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(4) $w-z=0$ $[1,3,4,5]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) $-x-y-z-2w=0$ $[2,3,4,5]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$2-faces$

(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
[1,2,3]	[1,2,4]	[1,3,4]	[2,3,4]	[1,2,5]	[1,3,5]	[2,3,5]	[1,4,5]	[2,4,5]	[3,4,5]

$Edges$

(1,2,3)	(1,2,4)	(1,2,5)	(1,3,4)	(1,3,5)	(1,4,5)	(2,3,4)	(2,3,5)	(2,4,5)	(3,4,5)
[1,2]	[1,3]	[2,3]	[1,4]	[2,4]	[3,4]	[1,5]	[2,5]	[3,5]	[4,5]

$Points$

(1,2,3,4) [1] $\begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$ (1,2,3,5) [2] $\begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ \frac{-3}{5} \\ 1 \end{bmatrix}$ (1,2,4,5) [3] $\begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \\ \frac{-2}{5} \\ \frac{-2}{5} \\ 1 \end{bmatrix}$ (1,3,4,5) [4] $\begin{bmatrix} \frac{4}{5} \\ \frac{-1}{5} \\ \frac{-1}{5} \\ \frac{-1}{5} \\ 1 \end{bmatrix}$ (2,3,4,5) [5] $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Any relation of Γ_4, in general, can be described by walking from $\mathcal{F}_4$ through its image domains by crossing side facets and returning to $\mathcal{F}_4$.

Table 4.2

Relations to 2-face classes in fundamental domain $\mathcal{F}_4$

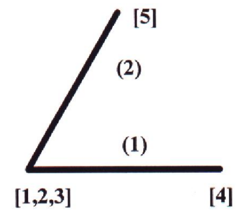
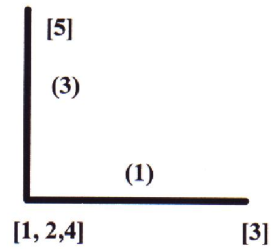
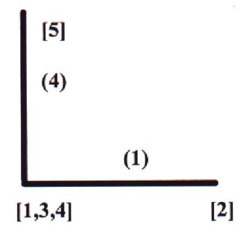
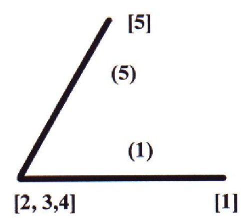
2-face class	*Relator transform, exponent*	*Diagram of 2-domain*
$(1,2)$ $[1,2,3]$	$\mu_1 \cdot \mu_2 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{12} = 3$	
$(1,3)$ $[1,2,4]$	$\mu_1 \cdot \mu_3 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{13} = 2$	
$(1,4)$ $[1,3,4]$	$\mu_1 \cdot \mu_4 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{14} = 2$	
$(1,5)$ $[2,3,4]$	$\mu_1 \cdot \mu_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{15} = 3$	

Table 4.2 (cont.1)

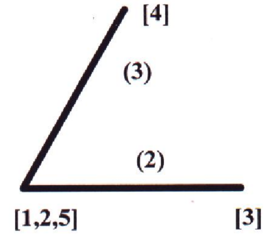
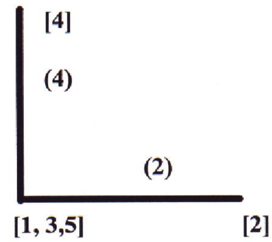
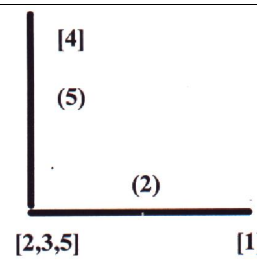
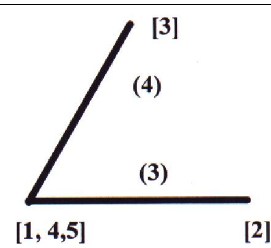
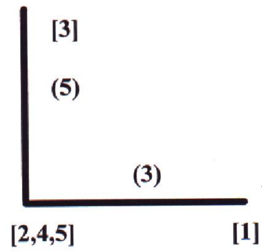
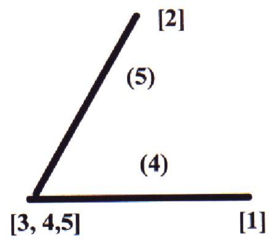
<table>
<tr><th>2-face class</th><th>Relator transform, exponent</th><th>Diagram of 2-domain</th></tr>
<tr><td>$(2,3)$
$[1,2,5]$</td><td>$$\mu_2 \cdot \mu_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\nu_{23} = 3$</td><td>
</td></tr>
<tr><td>$(2,4)$
$[1,3,5]$</td><td>$$\mu_2 \cdot \mu_4 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\nu_{24} = 2$</td><td>
</td></tr>
<tr><td>$(2,5)$
$[2,3,5]$</td><td>$$\mu_2 \cdot \mu_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\nu_{25} = 2$</td><td>
</td></tr>
<tr><td>$(3,4)$
$[1,4,5]$</td><td>$$\mu_3 \cdot \mu_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\nu_{34} = 3$</td><td>
</td></tr>
</table>

Table 4.2 (cont.2)

2-face class	*Relator transform,exponent*	*Diagram of 2-domain*
$(3,5)$ $[2,4,5]$	$\mu_3 \cdot \mu_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{35} = 2$	
$(4,5)$ $[3,4,5]$	$\mu_4 \cdot \mu_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\nu_{45} = 3$	

5 Fundamental domain for space group XXII.31/07/02

XXII.31/07/02 is the richest space group for the SN centred lattice. The point group Γ_{07} of space group $31/07/02 = \Gamma_7$ has 240 transforms, and can be generated by

(5.1)
$$\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{10} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma_6 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

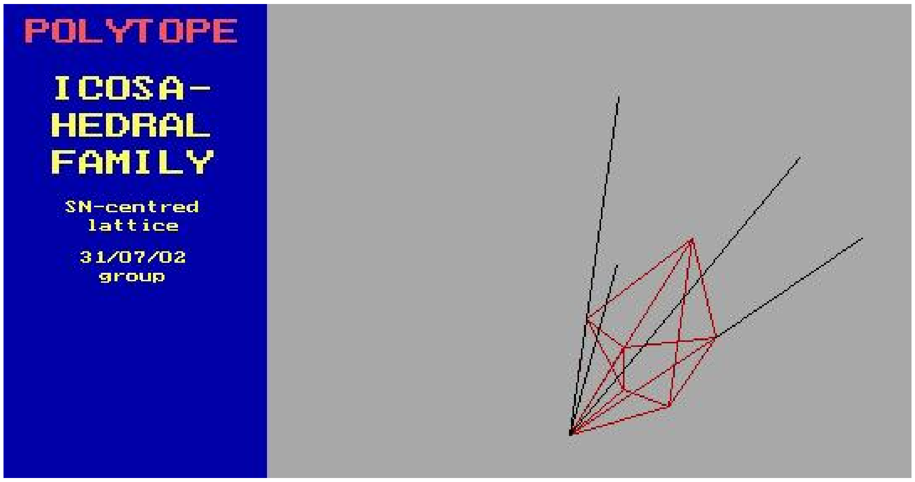


Figure 5.1

The fundamental domain $\mathcal{F}_7$ of space group Γ_7 has **7** vertices and **6** bisector 3-faces: one bisector is to the origin and the lattice point $(1;0;0;0)$, further **5** 3-faces appertain to the pyramidal $\mathcal{F}_{07}$.

Table 5.1

Pairing on fundamental domain $\mathcal{F}_7$

Source 3-plane and its points	*Pairing transform*	*Image 3-plane and its points*
$2x+y+z+w-1=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(\frac{1}{2};0;0;0)$	τ_1 $\begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$2x+y+z+w-1=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(\frac{1}{2};0;0;0)$
$y-x=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(0;0;0;0)$	τ_2 $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$y-x=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(0;0;0;0)$
$z-y=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(\frac{1}{2};0;0;0)$ $(0;0;0;0)$	τ_3 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$z-y=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(\frac{1}{2};0;0;0)$ $(0;0;0;0)$
$w-z=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(\frac{1}{2};0;0;0)$ $(0;0;0;0)$	τ_4 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$w-z=0$ $(\frac{1}{5};\frac{1}{5};\frac{1}{5};\frac{1}{5})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(\frac{1}{2};0;0;0)$ $(0;0;0;0)$
$-x-y-z-2w=0$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(0;0;0;0)$	τ_5 $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-x-y-z-2w=0$ $(\frac{2}{5};\frac{2}{5};\frac{2}{5};\frac{-3}{5})$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(0;0;0;0)$
$-y-z-w=0$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(\frac{1}{2};0;0;0)$ $(0;0;0;0)$	ρ $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$-y-z-w=0$ $(\frac{1}{2};\frac{1}{2};0;\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{4};\frac{1}{4};\frac{-1}{2})$ $(\frac{1}{2};\frac{1}{2};\frac{-1}{4};\frac{-1}{4})$ $(\frac{1}{2};0;0;0)$ $(0;0;0;0)$

Flag structure of fundamental domain $\mathcal{F}_7$

3 − faces

(1) $2x+y+z+w-1=0$ [1, 2, 3, 4, 5, 6]
$$\begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(2) $y-x=0$ [1, 2, 3, 4, 7]
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) $z-y=0$ [1, 2, 5, 6, 7]
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(4) $w-z=0$ [1, 3, 5, 7]
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) $-x-y-z-2w=0$ [2, 4, 6, 7]
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(6) $-y-z-w=0$ [3, 4, 5, 6, 7]
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2 − faces

(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,3)	(2,4)	(2,5)	(2,6)	(3,4)	(3,5)	(3,6)	(4,6)	(5,6)
[1,2,3,4]	[1,2,5,6]	[1,3,5]	[2,4,6]	[3,4,5,6]	[1,2,7]	[1,3,7]	[2,4,7]	[3,4,7]	[1,5,7]	[2,6,7]	[5,6,7]	[3,5,7]	[4,6,7]

Edges

(1,2,3)	(1,2,4)	(1,2,5)	(1,2,6)	(1,3,4)	(1,3,5)	(1,3,6)	(1,4,6)	(1,5,6)	(2,3,4)	(2,3,5)	(2,4,6)	(2,5,6)	(3,4,6)	(3,5,6)
[1,2]	[1,3]	[2,4]	[3,4]	[1,5]	[2,6]	[5,6]	[3,5]	[4,6]	[1,7]	[2,7]	[3,7]	[4,7]	[5,7]	[6,7]

Points

[1] (1,2,3,4) $\begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$
[2] (1,2,3,5) $\begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ \frac{-3}{5} \\ 1 \end{bmatrix}$
[3] (1,2,4,6) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{4} \\ \frac{-1}{4} \\ 1 \end{bmatrix}$
[4] (1,2,5,6) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{-1}{2} \\ 1 \end{bmatrix}$
[5] (1,3,4,6) $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
[6] (1,3,5,6) $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{-1}{2} \\ 1 \end{bmatrix}$
[7] (2,3,4,5,6) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Table 5.2

Relations to 2-face classes in fundamental domain $\mathcal{F}_7$

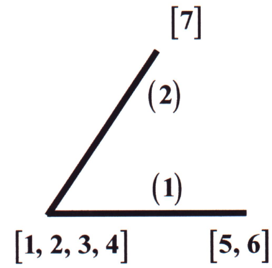
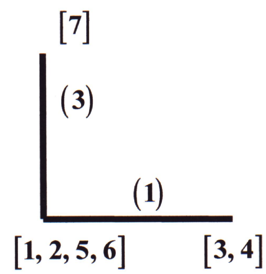
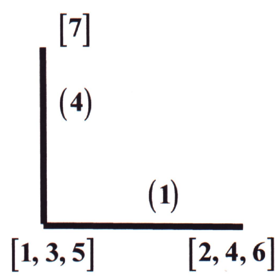
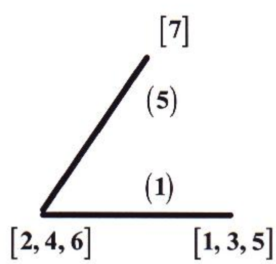
<table>
<tr><th>2-face class</th><th>Relator transform, exponent</th><th>Diagram of 2-domain</th></tr>
<tr><td>$(1,2)$
$[1,2,3,4]$</td><td>$$\tau_1 \cdot \tau_2 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ $$\nu_{12} = 3$$</td><td>
</td></tr>
<tr><td>$(1,3)$
$[1,2,5,6]$</td><td>$$\tau_1 \cdot \tau_3 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ $$\nu_{13} = 2$$</td><td>
</td></tr>
<tr><td>$(1,4)$
$[1,3,5]$</td><td>$$\tau_1 \cdot \tau_4 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ $$\nu_{14} = 2$$</td><td>
</td></tr>
<tr><td>$(1,5)$
$[2,4,6]$</td><td>$$\tau_1 \cdot \tau_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ $$\nu_{15} = 3$$</td><td>
</td></tr>
</table>

Table 5.2 (cont.1)

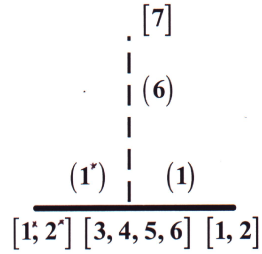
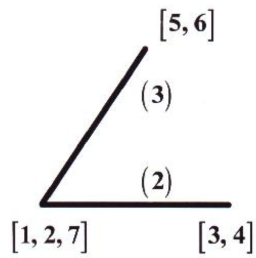
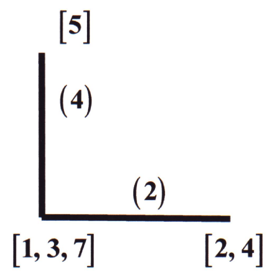
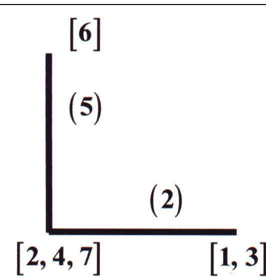
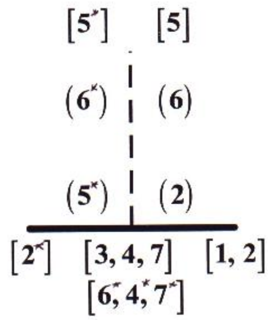
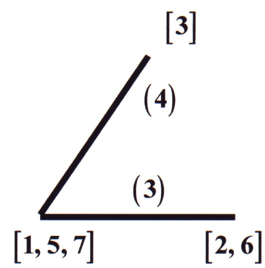
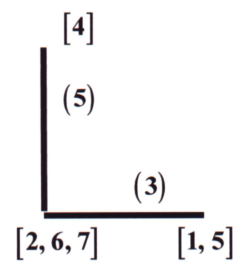
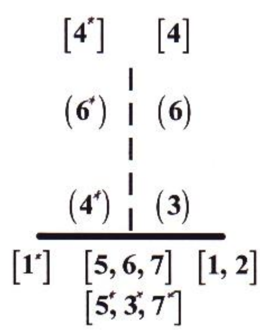
<table>
<tr><th>2-face class</th><th>Relator transform, exponent</th><th>Diagram of 2-domain</th></tr>
<tr><td>$(1,6)$
$[3,4,5,6]$</td><td>$$\rho\cdot\tau_1=\begin{pmatrix}-1&0&0&0&1\\0&0&0&-1&0\\0&0&-1&0&0\\0&-1&0&0&0\\0&0&0&0&1\end{pmatrix}$$ $$2\nu_1=2$$</td><td>
</td></tr>
<tr><td>$(2,3)$
$[1,2,7]$</td><td>$$\tau_2\cdot\tau_3=\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$ $$\nu_{23}=3$$</td><td>
</td></tr>
<tr><td>$(2,4)$
$[1,3,7]$</td><td>$$\tau_2\cdot\tau_4=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}$$ $$\nu_{24}=2$$</td><td>
</td></tr>
<tr><td>$(2,5)$
$[2,4,7]$</td><td>$$\tau_2\cdot\tau_5=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\-1&-1&-1&-1&0\\0&0&0&0&1\end{pmatrix}$$ $$\nu_{25}=2$$</td><td>
</td></tr>
</table>

Table 5.2 (cont.2)

<table>
<tr><th>2-face class</th><th>Relator transform,exponent</th><th>Diagram of 2-domain</th></tr>
<tr><td>$(2,6)$
$[3,4,7]$
$\downarrow \rho$
$(5^*,6^*)$
$[4^*,6^*,7^*]$</td><td>$$\rho \cdot \tau_2 = \tau_5 \cdot \rho = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\rho \cdot \tau_5 \cdot \rho \cdot \tau_2 = 1, \quad \nu_2 = 1 = \nu_5$</td><td>
</td></tr>
<tr><td>$(3,4)$
$[1,5,7]$</td><td>$$\tau_3 \cdot \tau_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\nu_{34} = 3$</td><td>
</td></tr>
<tr><td>$(3,5)$
$[2,6,7]$</td><td>$$\tau_3 \cdot \tau_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\nu_{35} = 2$</td><td>
</td></tr>
<tr><td>$(3,6)$
$[5,6,7]$
$\downarrow \rho$
$(4^*,6^*)$
$[3^*,5^*,7^*]$</td><td>$$\rho \cdot \tau_3 = \tau_4 \cdot \rho = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$\rho \cdot \tau_4 \cdot \rho \cdot \tau_3 = 1, \quad \nu_3 = 1 = \nu_4$</td><td>
</td></tr>
</table>

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