Reciprocity Principle for the Fundamental Solution of the Maxwell Equations

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Abstract

We examine the form of the Reciprocity Principle for the tensor fundamental solution of the three-dimensional Maxwell equation. A tensor fundamental solution is a generalisation of the Green function in case of the vector equation.

The Reciprocity Principle is demonstrated through a solution of the Maxwell equations in a stratified medium.

1 Introduction. Fundamental Solution

The propagation of the electromagnetic waves is described by the system of the Maxwell equations. The Maxwell equations formulated in frequency variables in the three-dimensional Euclidean space $\Re^3$ are as follows ([1])

$$\operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}, \quad \operatorname{rot}\mathbf{H} = -i\omega\varepsilon'\mathbf{E} + \mathbf{j}, \quad \varepsilon' = \varepsilon + i\sigma/\omega, \tag{1}$$

where $\mathbf{E}$, $\mathbf{H}$ are the vectors of the electric and magnetic fields, correspondingly. Parameter ω is frequency, μ, ε, σ denote the permeability (in vacuum), permittivity and conductivity. The source function $\mathbf{j}(\mathbf{R})$ is a given vector function with bounded support V_0. On V_0, $\mathbf{j}$ is assumed to be a (piecewise) continuous function or a set of dipols.

If μ is constant, the system (1) can be rewritten in the form of a single equation for $\mathbf{E}$:

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = k^2\mathbf{E} + i\omega\mu\mathbf{j}, \quad k^2 = \omega^2\mu\varepsilon', \quad Im(k) \geq 0. \tag{2}$$

We examine the case where μ is a positive constant, ε and σ are non-negative functions.

We suppose that $\Re^3$ can be decomposed into a finite number of domains G_j ($G_j \cap G_i = 0$ if $j \neq i$), in which σ is either a positive twice differentiable function or $\sigma \equiv 0$.

Note that equation (2) is an analogue of the Helmholtz equation in the case of three–dimensional vector equation with complex coefficients.

The boundaries S_j of the domains G_j are supposed to be surfaces of Lyapunov type (or S_j consist of a finite number of such surfaces). On the surfaces S_j where the coefficients ε' and k are discontinuous, appropriate compatibility conditions have to be prescribed. We assume for the tangential components of the vectors $\mathbf{E}$, $\mathbf{H}$, that

$$E_\tau,\ H_\tau\ (or\ (\operatorname{rot}\mathbf{E})_\tau)\ \ are\ continuous\ along\ S_j. \tag{3}$$

Furthermore, let V_R be a ball with large radius R_g and boundary S_R which includes all bounded domains G_j, and the sphere S_R crosses every infinite surface S_j . In the parts of S_R where $\sigma \neq 0$, the following estimates hold ([2]):

$$\lim_{R_g\to\infty} R_g \mid \mathbf{E} \mid = 0, \qquad \lim_{R_g\to\infty} R_g \mid \mathbf{H} \mid = 0. \tag{4}$$

If $\sigma \equiv 0$ in the domain crossed by S_R, then the Silver–Müller conditions can be used on an appropriate part of S_R ([2, 8, 13]):

$$\mid \mathbf{E} \mid = O(R_g^{-1}), \quad \mid \mathbf{H} \mid = O(R_g^{-1}), \quad \mid \mathbf{E} + \sqrt{\mu/\varepsilon}(\mathbf{n}\times\mathbf{H}) \mid = o(R_g^{-1}), \tag{5}$$

where $\mathbf{n}$ is the outward unit vector, normal to S_R. Note that conditions (3) and (4),(5) are usual in physics and are the consequences of the Conservation Laws.

Let us define a fundamental solution of the equation (2). For the vector equation, a fundamental solution is a tensor $\mathcal{E}$ (in Cartesian coordinates):

$$\mathcal{E}(\mathbf{R},\mathbf{R}_0) = \begin{pmatrix} E_x^x & E_x^y & E_x^z \\ E_y^x & E_y^y & E_y^z \\ E_z^x & E_z^y & E_z^z \end{pmatrix}, \tag{6}$$

where, for example, the column $\mathbf{E}^x(\mathbf{R},\mathbf{R}_0) = (E_x^x, E_y^x, E_z^x)^T$ is solution of the vector equation (2), that is, the electric field vector when the source is a unit dipole in the point $\mathbf{R}_0$ in x direction:

$$\mathbf{j}(\mathbf{R}) = (\delta(\mathbf{R}-\mathbf{R}_0), 0, 0)^T, \tag{7}$$

where $\delta(\mathbf{R}-\mathbf{R}) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ is the Dirac delta-function.

The tensor $\mathcal{E}$ satisfies the equation

$$\operatorname{rot}\operatorname{rot}\mathcal{E} = k^2\mathcal{E} + i\omega\mu\mathcal{D}, \qquad \mathcal{D} = \delta(\mathbf{R}-\mathbf{R}_0)\mathcal{I}, \tag{8}$$

where $\mathcal{I}$ is the unit tensor. $\operatorname{rot}\operatorname{rot}\mathcal{E}$ means the tensor the components of which are obtained by applying the operator $\operatorname{rot}\operatorname{rot}$ to the columns of $\mathcal{E}$: $\operatorname{rot}\operatorname{rot}\mathbf{E}^{\mathbf{x}}$, etc. For the vectors–columns of $\mathcal{E}$, the conditions (3) and (4),(5) are supposed to hold.

The fundamental solution is a generalisation of the Green function in the case of a vector equation. The fundamental solution is known as the Green tensor as well.

If the fundamental solution is known, the solution of the equation (2) for the arbitrary source function $\mathbf{j}(\mathbf{R}),\ \mathbf{R} \in V_0$ can be calculed as a quadrature ([4]):

$$\mathbf{E}(\mathbf{R}) = \fint_{\Re^3} \mathcal{E}(\mathbf{R},\mathbf{R}_0)\mathbf{j}(\mathbf{R}_0)d\mathbf{R}_0 - \frac{i\omega\mu}{3k^2(\mathbf{R})}\mathbf{j}(\mathbf{R}),$$

where by integral $\fint$ we denote the Cauchy's principal value of a singular integral. Note that components of the tensor $\mathcal{E}$ are singular functions at the pole $\mathbf{R}_0$ of the order of $O(|\mathbf{R}-\mathbf{R}_0|^{-3})$ ([4]).

The fundamental solution is important in the theory of the integral equation method, where the solution of the Maxwell vector equation in an infinite domain with local inhomogenuous subdomains can be reduced to that of the singular integral equation in a bounded domain ([5, 7]). A remarkeble advantage of the integral equation method proposed here is that it reduces the dimension of the problem and it is quite efficient when using parallel algorithms for the numerical solution of the integral equations, such as preconditioned domain decomposition algorithms (see for example [9, 10]).

2 The Reciprocity Principle for the Fundamental Solution

The Reciprocity Principle (or Reciprocity Law) is a theorem on the connection betweeen the electromagnetic fields $\mathbf{E}_1, \mathbf{H}_1$ and $\mathbf{E}_2, \mathbf{H}_2$ which are induced by electrical and magnetic sources $\mathbf{I}_1, \mathbf{I}_{\mu 1}$ and $\mathbf{I}_2, \mathbf{I}_{\mu 2}$ with the same frequence ω. The first and general form of the Reciprocity Principle is given by the Lorentz Lemma:

$$\mathrm{div}(\mathbf{E}_1 \times \mathbf{H}_2) - \mathrm{div}(\mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{I}_1\mathbf{E}_2 - \mathbf{I}_2\mathbf{E}_1 + \mathbf{I}_{\mu 2}\mathbf{H}_1 - \mathbf{I}_{\mu 1}\mathbf{H}_2.$$

This Lemma follows from the Maxwell equations if the space is characterized by the parameters ε and μ only. These parameters are supposed to be complex scalar functions or symmetrical tensors.

In an integral form, the Lemma can be rewritten as

$$\int_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) d\mathbf{S} = \int_V (\mathbf{I}_1\mathbf{E}_2 - \mathbf{I}_2\mathbf{E}_1 + \mathbf{I}_{\mu 2}\mathbf{H}_1 - \mathbf{I}_{\mu 1}\mathbf{H}_2) dV.$$

The integral form is more suitable if the sources are finite sums of dipols.

This representation can be used to infinite domain $\Re^3$ (as supposed in this paper). From the conditions at the infinity (4),(5), it follows that the integral in the left-hand side is equal to zero. Further, correspondingly to our suppositions, magnetic sources $I_{\mu 1}$ and $I_{\mu 2}$ are absent, the electric sources are assumed to have bounded supports.

For the fundamental solution of the electric field we use the following form ([12]).

Let $\mathbf{j}_1(\mathbf{R})$ be a distribution of sources in the domain V_1, and let the waves induced by these sources be denoted by $\mathbf{E}_1(\mathbf{R})$. Similarly, let the waves induced by the distribution of sources $\mathbf{j}_2(\mathbf{R})$ be denoted by $\mathbf{E}_2(\mathbf{R})$ in the domain V_2. From the Repciprocity Principle it follows that:

$$\int_{V_1} \mathbf{E}_2(\mathbf{R})\mathbf{j}_1(\mathbf{R}) dR = \int_{V_2} \mathbf{E}_1(\mathbf{R})\mathbf{j}_2(\mathbf{R}) d\mathbf{R}. \tag{9}$$

We use this formula in the following theorem.

Theorem 1. *For fundamental solution $\mathcal{E}$, the following Reciprocity Principle (or Reciprocity Law) holds*:

$$\mathcal{E}^T(\mathbf{R}_1, \mathbf{R}_2) = \mathcal{E}(\mathbf{R}_2, \mathbf{R}_1). \tag{10}$$

Proof. Let $\mathbf{E}^1$ be the vector of the electrical field in the point $\mathbf{R}$ induced by the dipole in the point $\mathbf{R}_1$, i.e. the source is the first column of the tensor $\mathcal{D}$ (8): $(\delta(\mathbf{R} - \mathbf{R}_1), 0, 0)^T$. Observe that $\mathbf{E}^1$ is the first column of the tensor $\mathcal{E}(\mathbf{R}, \mathbf{R}_1)$. Analogously, let $\mathbf{E}^2$ be the vector in the point $\mathbf{R}$ induced by dipole in the point $\mathbf{R}_2$ which is the second column of the tensor $\mathcal{D} : (0, \delta(\mathbf{R} - \mathbf{R}_2), 0)^T$. Then $\mathbf{E}^2$ is the second column of the tensor $\mathcal{E}(\mathbf{R}, \mathbf{R}_2)$.

Let $\mathbf{R}_1 \in V_1$ and $\mathbf{R}_2 \in V_2$, where V_1 and V_2 are arbitrary volumes. Then

$$\int_{V_1} \mathbf{E}^1(\mathbf{R}, \mathbf{R}_1)(0, \delta(\mathbf{R} - \mathbf{R}_2), 0)^T d\mathbf{R} = \int_{V_2} \mathbf{E}^2(\mathbf{R}, \mathbf{R}_2)(\delta(\mathbf{R} - \mathbf{R}_1), 0, 0)^T d\mathbf{R}.$$

From this equality, it follows (because of the properties of the delta -function) that:

$$E_2^1(\mathbf{R}_2, \mathbf{R}_1) = E_1^2(\mathbf{R}_1, \mathbf{R}_2).$$

It is obvious that same equalities can be obtained for the other components of $\mathcal{E}$ as well. From this equality the equation (10) follows directly. •

In another situation, a similar result has been obtained in [14].

Note that an analogous result can be obtained for fundamental solution of the magnetic fields $\mathcal{H}(\mathbf{R},\mathbf{R}_0)$.

3 Solution of the Maxwell Equation for a stratified medium and the Reciprocity Principle

Let us show the Reciprocity Principle for a fundamental solution through the following example.

The medium is supposed to be stratified: $k = k(z)$ ([6, 11]),

$$k(z) = \begin{cases} k_m(z),\ z_{m-1} < z < z_m,\ m = 1, 2, \dots n-1; \\ k_0 = const,\ z < z_0 = 0; \\ k_n = const,\ z_{n-1} < z; \end{cases} \tag{11}$$

where $k_m,\ m = 1, \dots n-1$ are continuously differentiable functions.

As pointed out earlier, the elements of the fundamental solution $\mathcal{E}(\mathbf{R},\mathbf{R}_0)$ have a nonintegrable singularity: $\|\mathcal{E}\| = O(|\mathbf{R}-\mathbf{R}_0|^{-3})$. It is useful to introduce a tensor–potencial $\mathcal{A}$:

$$\mathcal{H} = \operatorname{rot}\mathcal{A},\quad \mathcal{E} = i\omega\mu\left(\mathcal{A} + \nabla\left(\frac{1}{k^2}\operatorname{div}\mathcal{A}\right)\right). \tag{12}$$

The tensor $\mathcal{A}$ satisfies the equation

$$\operatorname{rot}\operatorname{rot}\mathcal{A} - k^2\nabla\left(\frac{1}{k^2}\operatorname{div}\mathcal{A}\right) - k^2\mathcal{A} = \mathcal{D}. \tag{13}$$

Note that $\|\mathcal{A}\| = O(|\mathbf{R}-\mathbf{R}_0|^{-1})$ ([4]).

First, let us introduce the vector potential $\mathbf{A}$ for $\mathbf{E}$ (2) as the solution of the equation

$$\operatorname{rot}\operatorname{rot}\mathbf{A} - k^2\nabla\left(\frac{1}{k^2}\operatorname{div}\mathbf{A}\right) - k^2\mathbf{A} = \mathbf{j}, \tag{14}$$

$$\mathbf{j} = (m_x\delta(\mathbf{R}-\mathbf{R}_0), m_y\delta(\mathbf{R}-\mathbf{R}_0), m_z\delta(\mathbf{R}-\mathbf{R}_0))^T.$$

On the surfaces of discontinuity z_i of $k(z)$, the components of $\mathbf{A}$ satisfy the following conditions:

$$[A_x]_{z_i} = [A_y]_{z_i} = [A_z]_{z_i} = 0;\ where\ [f]_{z_i} := f(z_i+0) - f(z_i-0); \tag{15}$$

$$\left[\frac{\partial A_x}{\partial z}\right]_{z_i} = \left[\frac{\partial A_y}{\partial z}\right]_{z_i} = \left[\frac{1}{k^2}\operatorname{div}\mathbf{A}\right]_{z_i} = 0.$$

At infinity:

$$\lim_{R\to\infty} R\,|\,\mathbf{A}\,| = 0,\quad if\quad k \neq 0\quad for\quad R\to\infty \tag{16}$$

$$|\,\mathbf{A}\,| = O(R_g^{-1}),\quad \frac{\partial}{\partial R}|\,\mathbf{A}\,| - ik\,|\,\mathbf{A}\,| = o(R^{-1}),\quad if\quad k\to 0.$$

Since k depends on z only, and $\Delta = \nabla\operatorname{div} - \operatorname{rot}\operatorname{rot}$, the vector equation (14) can be rewritten in a scalar form for components of $\mathbf{A}$ as

$$\Delta A_j + k^2 A_j = -m_j\delta(\mathbf{R}-\mathbf{R}_0),\quad j = x, y;$$

$$k^2 \operatorname{div}\left(\frac{1}{k^2}\nabla A_z\right) + k^2 A_z = -K(z)\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right) - m_z\delta(\mathbf{R}-\mathbf{R}_0), \tag{17}$$

$$K(z) = k^2 \frac{d}{dz}\frac{1}{k^2}.$$

Let us introduce cylindrical coordinates:

$$(r,\phi,z), \quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

From the equations (17) and conditions (15), it can be seen that A_x and A_y are independent of ϕ. Therefore it is useful to apply some integral transformation. We use the Hankel transformation ([1, 3]):

$$A_j(r,z) = \frac{m_j}{2\pi}\int_0^\infty J_0(tr)u(z,t)tdt, \quad j = x,y; \tag{18}$$

where J_0 is the Bessel function of the first kind, and respectively:

$$u(z,t) = \frac{2\pi}{m_j}\int_0^\infty J_0(tr)A_j(z,t)rdr.$$

Since $\delta(\mathbf{R}-\mathbf{R}_0) = (1/r)\delta(r)\delta(\phi)\delta(z-z_0)$, after multiplying the equation for A_j (17) by $rJ_0(rt)$ and integrating we get an ordinary differential equation for $u(z,t)$, where t is a parameter:

$$u'' - \alpha^2 u = -\delta(z-z_0), \quad \alpha^2 = t^2 - k^2, \quad Re(\alpha) > 0. \tag{19}$$

$$[u]_{z_i} = [u']_{z_i} = 0, \quad z_i \neq z_0; \quad [u]_{z_0} = 0, \quad [u']_{z_0} = -1; \quad \lim_{z\to\infty}|u| = 0.$$

Once the solutions for A_x and A_y have been determined we can solve the equation for A_z.

Let $B_x,\ B_y$ and B_z be the solutions of the equations

$$k^2 \operatorname{div}\left(\frac{1}{k^2}\nabla B_j\right) + k^2 B_j = -K(z)A_j, \quad j = x,y; \tag{20}$$

$$[B_j]_{z_i} = 0, \quad \left[\frac{1}{k^2}\frac{\partial}{\partial z}B_j\right]_{z_i} = -\left[\frac{1}{k^2}\right]A_j, \quad \lim_{R\to\infty}|B_j| = O(R^{-2});$$

$$k^2 \operatorname{div}\left(\frac{1}{k^2}\nabla B_z\right) + k^2 B_z = -m_z\delta(\mathbf{R}-\mathbf{R}_0); \tag{21}$$

$$[B_z]_{z_i} = 0, \quad \left[\frac{1}{k^2}\frac{\partial}{\partial z}B_z\right]_{z_i} = 0, \quad \lim_{R\to\infty}|B_z| = O(R^{-2}).$$

It is easy to show that

$$A_z = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + B_z,$$

and at the same time $B_x,\ B_y,\ B_z$ are independent of ϕ. So let

$$B_j(r,z) = \frac{m_j}{2\pi}\int_0^\infty J_0(tr)w(z,t)tdt, \quad j = x,y. \tag{22}$$

For w we obtain:

$$k^2\left(\frac{1}{k^2}w'\right)' - \alpha^2 w = -K(z)u \tag{23}$$

$$[w]_{z_i} = 0, \quad \left[\frac{1}{k^2} w'\right]_{z_i} = -\left[\frac{1}{k^2}\right]_{z_i} u, \quad \lim_{z\to\infty} w = 0.$$

For B_z we have:

$$B_z(r,z) = \frac{m_z}{2\pi} \int_0^\infty J_0(tr) v(z,t) t dt, \tag{24}$$

and

$$k^2 \left(\frac{1}{k^2} v'\right)' - \alpha^2 v = -\delta(z - z_0), \tag{25}$$

$$[v]_{z_i} = [v]_{z_0} = 0, \quad \left[\frac{1}{k^2} v'\right]_{z_i} = 0, \quad \left[\frac{1}{k^2} v'\right]_{z_i} = -\left(\frac{1}{k^2}\right)(z \pm 0)$$

Now we get the formulae for $E_x,\ E_z$ as follows

$$E_x = \frac{i\omega\mu}{2\pi k^2} \left(\int_0^\infty J_0(tr) \left(k^2 m_x u - \left(m_x \frac{(x-x_0)^2}{r^2} + \right.\right.\right. \tag{26}$$

$$\left.\left. + \ m_y \frac{(x-x_0)(y-y_0)}{r^2} \right) (u + w') t^2 \right) t dt - \int_0^\infty J_1(tr) \left(m_z \frac{x - x_0}{r} t v' - \right.$$

$$\left.\left. - \left(m_x \frac{(x-x_0)^2 - (y-y_0)^2}{r^2} + m_y \frac{2(x-x_0)(y-y_0)}{r^2} \right) (u + w') \frac{t}{r} \right) t dt \right),$$

$$E_z = \frac{i\omega\mu}{2\pi k^2} \left(m_z \int_0^\infty J_0(tr) v t^3 dt - \int_0^\infty J_1(tr) \left(m_x \frac{x - x_0}{r} + \right.\right.$$

$$\left.\left. - m_y \frac{y - y_0}{r} \right) \cdot (u' + t^2 w) t^2 dt \right).$$

The formula for E_y is symmetrical to it for E_x.

Note that from this formulae it is easy to obtain the formulae for components of the tensor $\mathcal{E}$. For example, $\mathbf{E}^{\mathbf{x}} = (E_x^x, E_y^x, E_z^x)^T$ can be got from (26) when $m_x = 1,\ m_y = m_z = 0$, etc.

Now let us solve the equations for $u,\ v,\ w$ in the simple case of a two–layer medium. Let

$$k(z) = \begin{cases} 0, & z < 0, \\ k_1 = const, & 0 < z. \end{cases}$$

In this case the function $K(z) = 0$. The equations for $u,\ v,\ w$ can be solved analytically. First, let a pole be in the point $\mathbf{R}_1(x_1, y_1, z_1)$ with $z_1 = 0 + 0$.

$$u(z,t) = \begin{cases} \frac{1}{t+\alpha_1} e^{tz}, & z < 0 \\ \frac{1}{t+\alpha_1} e^{-\alpha_1 z}, & 0 < z \end{cases}; \quad w = \begin{cases} -\frac{1}{t(t+\alpha_1)} e^{tz}, & z < 0 \\ -\frac{1}{t(t+\alpha_1)} e^{-\alpha_1 z}, & 0 < z \end{cases}; \quad v \equiv 0.$$

The functions E_y^x and E_z^x in the point (x_2, y_2, z_2) are equal (if $m_x = 1,\ m_y = m_z = 0$)

$$E_y^x = -\frac{i\omega\mu}{2\pi k_1^2} \frac{(x_2 - x_1)(y_2 - y_1)}{r^2} \int_0^\infty \left(J_0(tr) - \frac{2}{tr} J_1(tr) \right) e^{-\alpha_1 z_2} t^2 dt,$$

$$E_z^x = \frac{i\omega\mu}{2\pi k_1^2} \frac{x_2 - x_1}{r} \int_0^\infty J_1(tr) e^{-\alpha_1 z_2} t^2 dt. \tag{27}$$

Now let a pole be in the point $\mathbf{R}_2(x_2, y_2, z_2)$ with $z_2 > 0$. We get:

$$u = \begin{cases} \frac{1}{t+\alpha} e^{tz - \alpha_1 z_2}, & z < 0, \\ \frac{1}{2\alpha_1} e^{-\alpha_1(z_2 - z)} - \frac{t - \alpha_1}{2\alpha_1(t+\alpha_1)} e^{-\alpha_1(z+z_2)}, & 0 < z < z_2, \\ \frac{1}{2\alpha_1} \left(1 - \frac{t-\alpha_1}{t+\alpha_1} e^{-2\alpha_1 z_2}\right) e^{-\alpha_1(z - z_2)}, & z_2 < z. \end{cases}$$

$$w = \begin{cases} -\frac{1}{t(t+\alpha_1)} e^{tz-\alpha_1 z_2}, & z < 0, \\ -\frac{1}{t(t+\alpha_1)} e^{-\alpha_1(z+z_2)}, & 0 < z. \end{cases};$$

$$v = \begin{cases} 0, & 0 < z, \\ \frac{1}{2\alpha_1}\left(e^{-\alpha_1(z_2-z)} - e^{-\alpha_1(z_2+z)}\right), & 0 < z < z_2, \\ \frac{1}{2\alpha_1}\left(e^{-\alpha_1(z-z_2)} - e^{-\alpha_1(z_2+z)}\right), & z_2 < z \end{cases}$$

If $m_x = m_z = 0,\ m_y = 1$ we get for E_x^y in the point $(x_1, y_1, z_1 = 0 + 0)$:

$$E_x^y = -\frac{i\omega\mu}{2\pi k_1^2}\frac{(x_1 - x_2)(y_1 - y_2)}{r^2}\int_0^\infty \left(J_0(tr) - \frac{2}{tr}J_1(tr)\right) e^{-\alpha_1 z_2} t^2 dt, \tag{28}$$

and for E_x^z if $m_x = m_y = 0,\ m_z = 1$:

$$E_x^z = -\frac{i\omega\mu}{2\pi k_1^2}\ \frac{x_1 - x_2}{r}\int_0^\infty J_1(tr) e^{-\alpha_1 z_2} t^2 dt. \tag{29}$$

It is clear that $E_x^y(\mathbf{R}_1, \mathbf{R}_2) = E_y^x(\mathbf{R}_2, \mathbf{R}_1)$ and $E_x^z(\mathbf{R}_1, \mathbf{R}_2) = E_z^x(\mathbf{R}_2, \mathbf{R}_1)$. So we obtain the equalities analogous to the Reciprocity Principle (10).

Above we used the fact that the solutions of the equations (19),(23),(25) in the case of a two–layer medium were obtained in an explicite (analytical) form. Note that the explicit solutions can be obtained for several number of layers: for three, fourth or more layers, if the coefficient is a piecewise constant function of z. The Reciprocity Principle is true in these cases as well.

The above results corroborate this principle for an important case of the infinite stratified domain, where the coefficients are variables at infinity. As noted above, the important relations follow from the Reciprocity Principle for the fundamental solution (Green tensor) of the vector Maxwell equations, and the use of this principle is an essential step in the rigorous justification of the integral equation method.

On the other hand the integral equation method is quite efficient when solving wave propagation problems in an infinite inhomogeneous medium with variable parameters at infinity.

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