Mesh independent superlinear convergence of the conjugate gradient method for discretized linear elliptic systems

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1 Introduction

The conjugate gradient method is a widespread method for solving symmetric linear equations, mainly by using some preconditioner operators a description can be found in [1]. Superlinear convergence has been proved for operator equations where the operator is a compact perturbation of the identity (see in [9]), mesh independence results had been proved in [5]. Our aim is to extend these results, so that we shall prove this property for some elliptic partial differential systems.

First we consider an abstract problem, with the superlinear mesh independent convergence property, proved in [5]. In the following section we consider an elliptic partial differential system and we prove that it has an abstract form that is discussed in Section 2. In the next section we state the main mesh independence results and finally we give a numerical solution method, actually finite element method, and we give our results on the testing of the proved theoretical results, with code written in Matlab.

2 An abstract problem

Let H be a separable Hilbert space and g an arbitrary element in H. Let us consider the linear equation

$$Bu = g, \tag{1}$$

with a linear operator B satisfying the following conditons:

(i) B has the form of $B = S + Q$, where S is densely defined (in our application unbounded), Q is bounded and both are linear self-adjoint operators of the Hilbert space H,

(ii) there exists $p > 0$: $\langle Su, u\rangle \geq p\|u\|^2$ $(u \in D(S))$,

(iii) $\langle Qu, u\rangle \geq 0$ $(u \in H)$,

(iv) the operator $S^{-1}Q$ defined on H_S is compact and of type Hilbert-Schmidt.

Remark 1.

· *A bounded linear operator $A \in B(H)$ is compact by definition if it maps the closed unit ball to a compact set. If A is compact then it has at most countably many nonzero eigenvalues each with a finite dimensional eigenspace.*

· *A compact operator A is a Hilbert-Schmidt operator if*

$$|||A|||^2 = \sum_{j=1}^{\infty} |\nu_j|^2 < \infty,$$

where $\nu_j (j = 1, \dots, \infty)$ are the eigenvalues of A. Then $|||A|||$ is the Hilbert-Schmidt norm of A, see in e.g. [10].

· *Here H_S denotes the completion of D with the energy scalar product $\langle u, v\rangle_S = \langle Su, v\rangle$. By assumption (ii) $H_S \subset H$. Since a semi-bounded symmetric operator is self-adjoint if and only if it is superjective (see in eg. [6]), hence the definition of the operator $S^{-1}Q$ makes sense.*

· *In the latter we shall also use the notation $\langle u, v\rangle_B = \langle Bu, v\rangle$, which also defines a scalar product on $(H_S, \langle\cdot,\cdot\rangle_S)$.*

We replace (1) by its preconditioned form $(I + S^{-1}Q)u = S^{-1}g$ which is equivalent to the following weak formulation:

$$\langle u, v\rangle_S + \langle Qu, v\rangle = \langle g, v\rangle \ (\forall v \in H_S), \tag{2}$$

which has a unique solution $u \in H_S$ by conditions (ii) and (iii).

Now equation (2) is solved numerically using Galerkin discretization.
Let $V = \text{span}\{\varphi_1, \ldots, \varphi_k\} \subset H_S$ be a given finite-dimensional subspace,

$$\mathbf{S} = \{\langle \varphi_i, \varphi_j\rangle_S\}_{i,j=1}^k \text{ and } \mathbf{Q} = \{\langle Q\varphi_i, \varphi_j\rangle\}_{i,j=1}^k$$

be the Gram matrices corresponding to S and Q. Seeking the solutions in V in the form of $u_V = \sum_{j=1}^k c_j\varphi_j$ we obtain a finite linear system

$$(\mathbf{S} + \mathbf{Q})c = \mathbf{b}, \tag{3}$$

with $\mathbf{c} = (c_1, \ldots, c_k)^T$ and $\mathbf{b} = \{\langle g, \varphi_j\rangle\}_{j=1}^k$. By the preceding assumptions the matrix $\mathbf{S} + \mathbf{Q}$ is symmetric and positive definite.

As we did it in the abstract case, we use the matrix $\mathbf{S}$ as a preconditioner for system (3). After preconditioning we obtain the system

$$(\mathbf{I} + \mathbf{S}^{-1}\mathbf{Q})\mathbf{c} = \widetilde{\mathbf{b}}, \tag{4}$$

where $\widetilde{\mathbf{b}} = \mathbf{S}^{-1}\mathbf{b}$, and $\mathbf{I}$ denotes the identity matrix on $\mathbb{R}^k$. Now we apply the conjugate gradient method for the solution of system (4).
This of course will be a preconditioned conjugate gradient method with the preconditioner $\mathbf{S}$. The following theorems are proved in [5].

Theorem 1. *[5] Let assumptions (i)-(iv) hold. Then*

$$\|\mathbf{S}^{-1}\mathbf{Q}\|_F \le \left|\left|\left| S^{-1}Q \right|\right|\right|,$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix ($\|A\|_F^2 = \text{trace}AA^T$), and $|||A|||$ denotes the Hilbert-Schmidt norm of a compact operator.

Corollary 1. *[5] The conjugate gradient method applied to system (4) yields the following estimate, with the notation $\mathbf{e}_n = \mathbf{c}_n - \mathbf{c}$ (the difference of the nth iteration $\mathbf{c}_n$ from the exact solution $\mathbf{c}$):*

$$\frac{\|\mathbf{e}_n\|}{\|\mathbf{e}_o\|} \le \left(\frac{3\left|\left|\left| S^{-1}Q \right|\right|\right|^2}{2n}\right)^{n/2},$$

if $n \in \mathbb{N}$ is even and $n \ge \left|\left|\left| S^{-1}Q \right|\right|\right|^2$. This estimate is independent of the subspace V used in Galerkin discretization.

3 Properties of symmetric elliptic systems

In this section we consider self-adjoint second order elliptic boundary value systems and their finite element discretizations. We prove that this problem has an abstract form that satisfies the assumptions in Section 2.

3.1 The linear elliptic system

Let $d \le 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain. We consider the elliptic problem

$$\begin{cases} -\text{div}(G_1\nabla u) + d_{11}u + d_{12}v = g_1 \\ -\text{div}(G_2\nabla v) + d_{21}u + d_{22}v = g_2 \\ u_{|\partial\Omega} = 0 \\ v_{|\partial\Omega} = 0 \end{cases} \tag{5}$$

under the following assumptions:

(a) $\partial\Omega$ is piecewise C^2 and Ω is locally convex at the corners,

(b) $G_1, G_2 \in C^1(\bar{\Omega}, \mathbb{R}^{d\times d})$, both symmetric at the points of $\bar{\Omega}$, and $\exists 0 < m \le M < \infty :$

$$m\|\xi\|^2 \le G_i(x)\xi\cdot\xi \le M\|\xi\|^2, \ \ (x\in\bar{\Omega},\ \xi\in\mathbb{R}^d,\ i=1,2),$$

(c) $D = D^T = \{d_{ij}\}_{i,j=1}^2 \in C(\bar{\Omega}, \mathbb{R}^{2\times 2})$ and $D \ge 0$ on $\bar{\Omega}$,

(d) $g_i \in L^2(\Omega)\ \ (i=1,2)$.

Remark 2.

- *The number of equations in (5) can be any positive integer r, for simplicity we have chosen $r = 2$.*
- *These type of equations may arise when linearizing a nonlinear system*

$$\begin{pmatrix} -\Delta & & \\ & \ddots & \\ & & -\Delta \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} + F(x, u_1, \dots, u_r) = \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}$$

in the case when F is continuously differentiable and has a potential (therefore by the Young-theorem the linearized equation is symmetric).

In the next two sections we shall prove that the system (5) corresponds to an operator equation of the form we mentioned in Section 2.

Let H be the product space $L^2(\Omega)\times L^2(\Omega)$ with the inner product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \langle u_1, v_1\rangle_{L^2(\Omega)} + \langle u_2, v_2\rangle_{L^2(\Omega)}.$$

The notation $\mathbf{u} = \binom{u_1}{u_2}$ will hold from now on. The definition of the operators S and Q are

$$S\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\mathrm{div}(G_1\nabla u_1) \\ -\mathrm{div}(G_2\nabla u_2) \end{pmatrix}, \ \ (\mathbf{u}\in D(S)), \tag{6}$$

and

$$Q\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = D\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ \ \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\in H\right), \tag{7}$$

where $D(S) = (H^2(\Omega)\cap H_0^1(\Omega))\times(H^2(\Omega)\cap H_0^1(\Omega))$. Therefore the system can be written as

$$(S+Q)\mathbf{u} = \mathbf{g}, \tag{8}$$

which has the form of (1).

3.2 Proving assumptions

The following easy computations show that the assumptions of Section 2 hold for the operator equation (8).

By the Ostrogradsky-Gauss theorem and the homogeneous Dirichlet boundary conditions

$$\left\langle S\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\rangle = -\int_\Omega (\mathrm{div}(G_1\nabla u)\cdot\nabla u + \mathrm{div}(G_2\nabla v)\cdot\nabla v) =$$
$$\int_\Omega (G_1\nabla u\cdot\nabla u + G_2\nabla v\cdot\nabla v),$$

hence with assumption (b) we have

$$m\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right) \le \left\langle S\binom{u}{v},\binom{u}{v}\right\rangle \le M\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right). \tag{9}$$

The Sobolev inequality gives

$$\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right) \ge \nu\int_\Omega \left(u^2+v^2\right), \left(\binom{u}{v}\in D(S)\right),$$

so we have that assumption (ii) is fulfilled with $p=\nu m$, and beyond this by (9) we also have that the energy space H_S coincides with the space $(H_0^1(\Omega))\times(H_0^1(\Omega))$.

It is obvious that assumption (iii) is fulfilled.

Since S is the pair of the symmetric operators $S_1, S_2 : L^2(\Omega)\to L^2(\Omega)$, which are symmetric and superjective by assumptions (a) and (b), therefore self-adjoints (see in [4]), hence their pair S is also self-adjoint. It is obvious that $D\in L^\infty(\Omega)$ in the sense that $d_\infty=\sup_{\bar\Omega}\|D(x)\|_2<\infty$. Hence assumption (i) also holds.

Let us take $\mathbf{u}\in H$ and $\mathbf{v}\in H_S$.

$$\begin{aligned}\left|\left\langle S^{-1}Q\binom{u_1}{u_2},\binom{v_1}{v_2}\right\rangle_S\right| = \left|\int_\Omega D\mathbf{u}\cdot\mathbf{v}\right| \le \\ d_\infty\int_\Omega\sqrt{(\|u_1\|^2+\|u_2\|^2)}\sqrt{(\|v_1\|^2+\|v_2\|^2)} \le \\ d_\infty\sqrt{\int_\Omega(\|u_1\|^2+\|u_2\|^2)}\sqrt{\int_\Omega(\|v_1\|^2+\|v_2\|^2)} = \\ d_\infty\left\|\binom{u_1}{v_2}\right\|_H\left\|\binom{v_1}{v_2}\right\|_H \le C\left\|\binom{u_1}{v_2}\right\|_H\left\|\binom{v_1}{v_2}\right\|_S,\end{aligned}$$

where $C=\frac{d_\infty}{\sqrt{m\nu}}$. From this we conclude that the operator $S^{-1}Q$ is bounded from H to H_S. Since the embedding $H_S\hookrightarrow H$ is compact, because it is the product of the embedding operator from the Sobolev space $H_0^1(\Omega)$ to $L^2(\Omega)$, therefore the operator $S^{-1}Q : H_S\to H_S$ is also compact.

So far we have proved all assumptions (i)-(iv) but the Hilbert-Schmidt property of $S^{-1}Q$, we remark that the existence and uniqueness of the weak solution of (5) can be easily verified on the usually way as in elliptic problems even with weaker assumptions to Ω or the matrices G_i, D.

3.3 The Hilbert-Schmidt property

For the required estimation of the norm $|||S^{-1}Q|||$, we use the variational property of the eigenvalues of a compact self-adjoint operator proved in e.g. [10].

Theorem 2. *Let X be a separable Hilbert-space and $A : X\to X$ a strictly positive self-adjoint compact operator, then*

$$\lambda_{j+1}=\sup_{x\perp s_1,\dots,s_j}\frac{\langle Ax,x\rangle}{\langle x,x\rangle}=\inf_{x_1,\dots x_j}\sup_{x\perp x_1,\dots,x_j}\frac{\langle Ax,x\rangle}{\langle x,x\rangle},$$

where λ_{j+1} denotes the $(j+1)$th eigenvalue of A, and $s_1,\dots,s_j$ are the eigenvectors corresponding to the first j eigenvalues.

With this theorem we can estimate the eigenvalues of $S^{-1}Q$. From now on λ_j denotes the jth eigenvalue of the operator $S^{-1}Q$. By equation (9) we have

$$\frac{\langle S^{-1}Q\mathbf{u},\mathbf{u}\rangle_S}{\langle\mathbf{u},\mathbf{u}\rangle_S}=\frac{\int_\Omega D\mathbf{u}\cdot\mathbf{u}}{\int_\Omega S\mathbf{u}\cdot\mathbf{u}}\le\frac{d_\infty}{m}\frac{\int_\Omega\mathbf{u}\cdot\mathbf{u}}{\int_\Omega|\nabla u_1|^2+|\nabla u_2|^2}, \tag{10}$$

where d_∞ is as defined before, so we have

$$\lambda_{j+1}=\inf_{\mathbf{u_1},\dots\mathbf{u_j}}\sup_{\mathbf{u}\perp\mathbf{u_1},\dots,\mathbf{u_j}}\frac{\langle S^{-1}Q\mathbf{u},\mathbf{u}\rangle_S}{\langle\mathbf{u},\mathbf{u}\rangle_S}\le\frac{d_\infty}{m}\lambda'_{j+1},$$

where

$$\lambda'_{j+1} = \inf_{\mathbf{u_1},\dots\mathbf{u_j}} \sup_{\mathbf{u}\perp\mathbf{u_1},\dots,\mathbf{u_j}} \frac{\int_\Omega \mathbf{u}\cdot\mathbf{u}}{\int_\Omega |\nabla u_1|^2 + |\nabla u_2|^2}.$$

Now we define the operators S' and Q' in the same way as we did it in Section 3.1, with $G'_1 \equiv G'_2 \equiv I$ and $D' \equiv I$, where I is the identity matrix of $\mathbb{R}^2$, defining the system

$$\begin{cases} -\mathrm{div}(\nabla u) + u = g_1 \\ -\mathrm{div}(\nabla v) + v = g_2 \\ u_{|\partial\Omega} = 0 \\ v_{|\partial\Omega} = 0. \end{cases} \tag{11}$$

We observe that by Theorem 2 λ'_{j+1} is the $(j+1)$th eigenvalue of ${S'}^{-1}Q'$. Since this system is decoupled, the operator ${S'}^{-1}Q'$ is the pair of the two operators $(-\Delta)^{-1}$, hence the eigenvalues of ${S'}^{-1}Q'$ are the same as the eigenvalues of $(-\Delta)^{-1}$, with doubled multiplicity.

It is known that if $\Omega \subset \Omega'$ then the eigenvalues μ_j of $(-\Delta)^{-1}$ has the following property (see in [8]): $\mu_j(\Omega) \le \mu_j(\Omega')$. Now take $\Omega' = R$ the rectangle that contains a translate of Ω. Since the eigenvalues of $(-\Delta)^{-1}$ are known on a rectangle (see in [8]) we can summarize the above results in the following theorem :

Theorem 3. *With the previous notations, the operator $S^{-1}Q$ is Hilbert-Schmidt if and only if $d \le 3$, and the following computable estimates are true for its Hilbert-Schmidt norm depending on space dimension:*

- $d = 1$, *take* $R = [0, a_1]$, *then*

$$\sigma_1^2 = |||S^{-1}Q|||^2 \le \frac{2d_\infty^2}{m^2\pi^2} \sum_{i=1}^{\infty} \left(\frac{i^2}{a_1^2}\right)^{-2}$$

- $d = 2$, *take* $R = [0, a_1] \times [0, a_2]$, *then*

$$\sigma_2^2 = |||S^{-1}Q|||^2 \le \frac{2d_\infty^2}{m^2\pi^2} \sum_{i,j=1}^{\infty} \left(\frac{i^2}{a_1^2} + \frac{j^2}{a_2^2}\right)^{-2}$$

- $d = 3$, *take* $R = [0, a_1] \times [0, a_2] \times [0, a_3]$, *then*

$$\sigma_3^2 = |||S^{-1}Q|||^2 \le \frac{2d_\infty^2}{m^2\pi^2} \sum_{i,j,k=1}^{\infty} \left(\frac{i^2}{a_1^2} + \frac{j^2}{a_2^2} + \frac{k^2}{a_3^2}\right)^{-2}.$$

Remark 3.

- *The assumptions in (5) can be more general as mentioned in [5].*
- *As mentioned before, the number of the equations in the system (5) may be any positive integer r. One may prove an analogous statement as in Theorem 3, replacing 2 by r.*

4 The mesh independence result for the finite element method

4.1 The formulation of the finite element method and the solution algorithm

Let $V_h^0 \subset H_0^1$ be a finite element subspace and define $V_h = V_h^0 \times V_h^0 \subset H_S$. We look for the approximate solution of (5) in V_h. That is we look for $\mathbf{u_h} \in V_h$ such that for all $\mathbf{w_h} \in V_h$

$$\int_\Omega (G_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + G_1 \nabla u_{h,1} \cdot \nabla w_{h,1} + D\mathbf{u_h} \cdot \mathbf{w_h}) = \int_\Omega (g_1 w_{h,1} + g_2 w_{h,2}),$$

which is equivalent to the following linear algebraic system:

$$(G_h + D_h)c = g_h, \tag{12}$$

where the stiffness and mass matrices G_h and D_h are defined as usual. We solve it by preconditioning with G_h. The algorithm is as follows:

1. Let $c_0 \in \mathbb{R}^k$ be arbitrary. Let $r_0 = c_0 + y_0 - g_h$, where y_0 is the solution of the preconditioner equation $G_h y_0 = D_h c_0$.
   Then set $p_0 = r_0$.

2. After we have c_n, r_n, p_n, define

$$\alpha_n = \frac{\langle G_h r_n, p_n \rangle}{\langle (G_h + D_h) p_n, p_n \rangle}, \quad c_{n+1} = c_n - \alpha_n p_n;$$

3. $r_{n+1} = r_n - \alpha_n (p_n - y_n)$, where y_n is the solution of the preconditioner equation $G_h y_n = D_h p_n$;

4. $$\beta_n = \frac{\langle (G_h + D_h) p_n, r_{n+1} \rangle}{\langle (G_h + D_h) p_n, p_n \rangle}, \quad p_{n+1} = r_{n+1} - \beta_n p_n.$$

Now we can state our main result:

Theorem 4. *The above described preconditioned conjugate gradient method applied to (12) has the following mesh independent estimation of its error:*

$$\frac{\|e_n\|_B}{\|e_0\|_B} \le \left(\frac{3}{2n} \sigma_d^2 \right)^{n/2},$$

where σ_d is defined in Theorem 3 and $n \ge (3/2)\sigma_d^2$.

4.2 Numerical experiments

Here we illustrate the preceding theoretical results by some numerical tests. The code was written in MATLAB. Even though the discretizations are the simplest, the stiffness and mass matrices were built and not generated by the tools of MATLAB. We also solved the linear equations with the solver of MATLAB, the error was computed as the $\mathbf{B}$-norm of the difference of the iteration solution and the solution given us by MATLAB.

The program is on the unit square $[0,1] \times [0,1]$, with $G_i \equiv I$ and D is some matrix satisfying assumption (b) of Section 3.1. Equidistant mesh and the canonical Courant elements were used, in the table h denotes the mesh width on $[0,1]$, the finite element mesh consists of $N = 2/h^2$ triangles. In the following tables σ_d^2 is calculated by Theorem 3 and

$$\sigma_d^{*2} = \sup_{n \ge (3/2)\sigma_d^2} \{ (\|e_n\|_B / \|e_0\|_B)^{2/n} (2n/3) \| \}$$

is the constant for the superlinear convergence. The results are:

h	1/10	1/20	1/30	1/40	1/50	1/60
σ_d^2	4.7619	5.2360	5.3852	5.4571	5.5.4994	5.5271
σ_d^{*2}	2.7016	2.6072	2.4889	2.5957	2.5275	2.4720

Table 1: estimation of the superlinear constant σ_d^2

The above tables show that the numerical example is consistent to our theoretical result as the PCG has the following empirical estimation:

$$\frac{\|e_n\|_B}{\|e_0\|_B} \le \left(\frac{3}{2n} \sigma_d^{*2} \right)^{n/2}.$$

5 Remarks on numerical realization

The advantage of this method, that it has good convergence, but preconditioning could take much time that in practice the convergence loses its superlinear property.
The preconditioning with the main part may be done by some fast solvers. Most of the fast solvers exploit the special shape of the domain Ω and the structure of the discretized equations. There are such fast solvers with cost $O(N \log N)$ see e.g. [2, 7]. Using multigrid-methods for preconditioning is also acceptable if the elliptic main part has constant coefficients, since then the computation of the stiffness matrices on the different levels are much easier.

Here we give a realization of the algorithm and we discuss the convergence in practice. In the algorithm in order to compute the numbers α_k, β_k, we have to compute the S-scalar products:

$$\langle r_k, p_k \rangle_{G_h}, \langle (I_h - G_h^{-1} D_h) p_k, p_k \rangle_{G_h}, \langle (I_h - G_h^{-1} D_h) r_{k+1}, p_k \rangle_{G_h},$$

and the vector $Q_h p_k$ and (with preconditioning) the vector $G_h^{-1} Q_h p_k$. Expressing these scalar-products we have that the following scalar products are to be computed (using the symmetricity of the occurrent matrices):

$$\langle r_k, G_h p_k \rangle, \langle G_h p_k, p_k \rangle, \langle D_h p_k, p_k \rangle, \langle r_{k+1}, G_h p_k \rangle, \langle r_{k+1}, D_h p_k \rangle.$$

So we need to calculate the matrix-vector products: $G_h p_k, D_h p_k$ (we need this last for the preconditoning also), this means $Nr \cdot 2l + Nr \cdot 2rl$ operations, where l is the maximum number of finite elements whose support intersects the support of a fixed one. Then we need to compute the 5 scalar products, this means $5 \cdot 2rN$. Now we summarize the constants, and define some new ones that we shall need:

- d: dimension of Ω $(d = 1, 2, 3)$,
- N: number of finite elements,
- h: the mesh width,
- r: number of equations (therefore we have a Nr dimensional linear algebraic system),
- l: maximum number of finite elements those support intersects to the support of a fixed one.

Now we give an intuitive approximation of the efficiency of our method. Since the discretization error is $O(h)$ (when using Courant elements), it's enough for us to solve the discrete equation within the bound $O(h)$. Therefore we need that $\|e_n\|_{S+Q} \leq O(h)$. So the number of operations is a function of the above listed quantities $n = n(d, N, h, r, l)$. We execute this computation for the example mentioned in Section 4.2, so $l = 6, N = 2/h^2, d = 2$.
In one step of the PCG we have to make $Nr \cdot 2l + Nr \cdot 2rl + 5 \cdot 2rN = 12Nr + 12Nr^2 + 10Nr = 12Nr + 22Nr$ operations computing the scalar products. In addition we have to make the preconditionings with a possible $rO(N \log N)$ operations. This means at most $const \cdot (Nr(1 + \log N) + Nr^2)$ operations. By Theorem 4 we need the following condition on the number of steps

$$\left(\frac{3}{2n} \sigma_d^2 \right)^{n/2} = O(h),$$

here σ_d is from our numerical exapmle. For fixed h we made an estimate on n using MATLAB. Then for fixed r we calculated the 'cost' of the PCG algorithm for $h = \frac{1}{50}, \ldots, \frac{1}{2500}$ and supposing that this is a power function of the unknown parameters, i.e.

$$n = c(rN)^\alpha \iff \log n = \log c + \alpha \log rN$$

we made an approximation on this α. The following two tables show the results whether an $O(N \log N)$ or an $O(N^2)$ preconditioning method would be used.

r	20	40	60	80	100
α	1.0080	1.0054	1.0029	1.0022	1.0018

Table 2: using an $O(N \log N)$ preconditioner

Our conclusion is that if a fast solver for the preconditioning exists, then this PCG method is an easily realiziable and fast method.

r	20	40	60	80	100
α	2.0043	2.0033	2.0013	2.0008	2.0005

Table 3: using an $O(N^2)$ preconditioner

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