On the Full Multigrid Basis Transformation in Dirichlet Domain Decomposition Preconditioners: Model Problem Analysis and Numerical Experiments

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Abstract

The domain decomposition (DD) technique is well-suited for constructing parallel partial differential equation solvers. By means of the Additive Schwarz Method we derive and analyse Dirichlet-type DD preconditioners. Such preconditioners contain three components which are the (modified) Schur complement preconditioner, the local Dirichlet problem preconditioner, and the so-called basis transformation. Of all components, the last one plays the most crucial role.

Amongst other methods, multigrid techniques have been popular for defining this basis transformation. However, the quality of the preconditioner deteriorates as $h \to 0$. In this paper we investigate whether the use of the full multigrid method can remedy this shortcoming.

The technical theoretical analysis has been carried out for a model problem and the full *two-grid* operator. In numerical experiments the full *multigrid* basis transformation has been tested. The analysis shows that full multigrid behaves asymptotically as the multigrid method but displays better numerical performance.

Keywords: Domain Decomposition, Preconditioners, Full Multigrid, Finite Elements, Conjugate Gradient Method, Parallel Algorithms

1 Introduction

The interest in parallel solvers for partial differential equations has risen in recent years. Thus the domain decomposition method (DD) became increasingly popular since it contains a natural parallel structure. As it is usual with this method, we utilize the resulting block system of linear equations

$$\begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix}$$

(cf. section 3 for a more detailed description). The Additive Schwarz Method has been employed to construct preconditioners C of the type

$$C := \begin{pmatrix} I_C & K_{CI}B_I^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ B_I^{-1}K_{IC} & I_I \end{pmatrix}$$

for the (parallel) conjugate gradient method (cf. [12, 13], and section 3). Two of the three components of this DD preconditioner, namely the (modified) Schur complement preconditioner C_C, and the local Dirichlet problem preconditioner C_I, have been studied intensively by the DD community (cf. the proceedings of the international Symposia on "DD methods for partial differential equations" since 1987 [7, 3, 4, 8, 18, 1, 19], and also [2, 5, 6]). Haase/Langer/Meyer [12, 13] proved that the quality of the DD preconditioner is dominantly influenced by the third component, the basis transformation B_I. This basis transformation determines the perturbation T_C of the Schur complement which, in turn, influences the spectral radius $\mu = \varrho(S_C^{-1}T_C)$ and finally the relative condition number of the preconditioner, $\kappa(C^{-1}K) = O(\mu)$ (provided that C_C and C_I are chosen appropriately).

Several ideas to choose the basis transformation B_I which is defined implicitly by some iteration method are summarized in section 3. Our interest is focused on the multigrid method for which it is known that

- the application of $s = O(\ln h^{-1})$ multigrid cycles ensures a bounded condition number $\kappa(C^{-1}K) = O(1)$ (cf. [12]).
- the application of $s = 1$ multigrid cycle results (in numerical experiments) in a growing condition number $\kappa(C^{-1}K) = O(h^{-1})$ as $h \to 0$ [12].

In this paper we investigate the use of one *full* multigrid cycle for the definition of the basis transformation. Special interest is paid firstly as to whether the increasing condition number $\kappa(C^{-1}K)$ can be overcome as $h \to 0$, and secondly to the computational expense.

Sections 2 is devoted to the model problems whereas section 3 deals with the DD preconditioner and the basis transformation B_I. The theoretical analysis of the full multigrid basis transformation is performed in section 4. Unfortunately the analysis turned out to be so technical that only the full *two-grid* operator on a simple model problem could be treated. In section 5 the numerical experiments with the full *multigrid* operator are given. Section 6 summarizes the results obtained.

2 The model problems and the domains

Here we consider exclusively the Poisson equation with homogeneous Dirichlet boundary conditions:

$$\text{Find } u(x) \in H_o^1(\Omega): \qquad \int_\Omega \nabla^T u \nabla v = \int_\Omega fv \qquad \forall v \in H_o^1(\Omega)$$

and Ω being a bounded two-dimensional domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. By means of the usual linear triangular finite element discretization we obtain a system of linear equations $K\,\underline{u} = \underline{f}$ with a symmetric and positive definite matrix K.

In this paper we consider two model problems. The *model problem 1* is given by the differential equation mentioned above and the domain Ω consisting of two unit squares. This domain is shown in Figure 1 which also illustrates some terms to be introduced later.

The model problem analysis is performed exclusively on this model problem 1. For numerical experiments we also use the *model problem 2*. The corresponding domain Ω and the subdomains Ω_i are depicted in Figure 2.

For a further (real life) problem the reader is referred to [20].

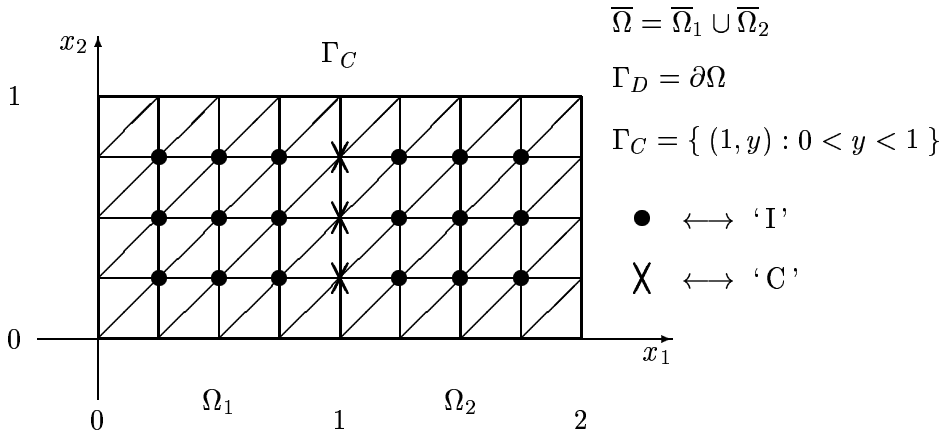

Figure 1: Model problem 1 (triangulation and subdomains)

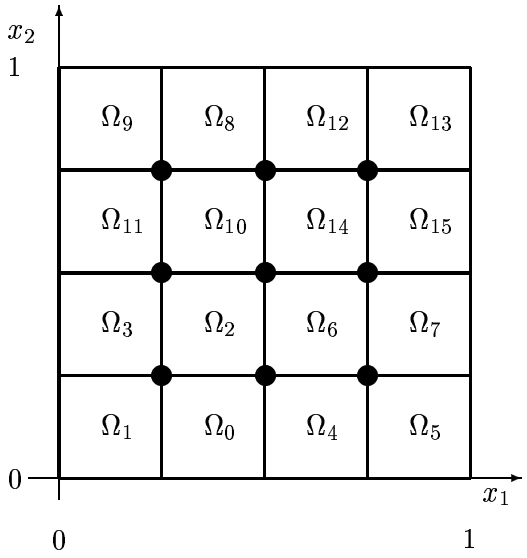

Figure 2: Model problem 2

3 The Domain Decomposition Preconditioner

For the domain decomposition method (FE substructure technique) let the domain Ω be subdivided into p non-overlapping subdomains Ω_i. The *coupling boundary* Γ_C is defined as $\Gamma_C := \bigcup_{i=1}^{p} \partial\Omega_i \setminus \Gamma_D$ with Γ_D being the Dirichlet boundary. The index 'C' always refers to nodes on that coupling boundary Γ_C whereas the index 'I' relates to nodes inside the subdomains Ω_i. Figure 1 may serve as an example.

For the finite element method used here we utilize the basis $\Phi = \{\varphi_1, \varphi_2 \ldots \varphi_N\}$ consisting of the common piecewise linear ansatz functions φ_i over the internal nodes of the triangulation. Let the numbering be such that all ansatz functions related to Γ_C come first, followed by those related to the inner nodes of Ω_1, Ω_2 ... With this special ordering the resulting system of linear equations can be written in block form

$$\begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix} \quad .$$

The basis transformation of the FE basis Φ to the *approximate discrete harmonic basis* $\widetilde{\Psi}$ is given by

$$\widetilde{\Psi} := \Phi * \begin{pmatrix} I_C & O \\ -B_I^{-1}K_{IC} & I_I \end{pmatrix} \quad .$$

The *Additive Schwarz Method (ASM)* corresponding to the splitting of the finite element space into the subspaces $\mathrm{span}\{\Phi * (I_C\,,\, -B_I^{-1}K_{IC})^T\}$ and $\mathrm{span}\{\Phi * (O\,,\, I_I)^T\}$ yields the preconditioner (cf. [12, 13, 23])

$$D = \begin{pmatrix} I_C & K_{CI}B_I^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} S_C + T_C & O \\ O & K_I \end{pmatrix} \begin{pmatrix} I_C & O \\ B_I^{-1}K_{IC} & I_I \end{pmatrix} \quad .$$

The matrices B_I, $S_C := K_C - K_{CI}K_I^{-1}K_{IC}$, $T_C := K_{CI}(K_I^{-1} - B_I^{-T})K_I(K_I^{-1} - B_I^{-1})K_{IC}$, and $S_C + T_C$ are referred to as the *basis transformation*, the *Schur complement*, the *perturbation of the Schur complement*, and the *modified Schur complement*, respectively.

Finally the matrices $S_C + T_C$ and K_I are replaced by symmetric, positive definite and spectrally equivalent matrices C_C and C_I, i.e.

$$\underline{\gamma}_C\, C_C \;\le\; S_C + T_C \;\le\; \overline{\gamma}_C\, C_C \quad \text{and} \quad \underline{\gamma}_I\, C_I \;\le\; K_I \;\le\; \overline{\gamma}_I\, C_I \quad .$$

It has been shown in [12, 13] that the resulting preconditioner

$$C := \begin{pmatrix} I_C & K_{CI}B_I^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ B_I^{-1}K_{IC} & I_I \end{pmatrix} \tag{1}$$

is spectrally equivalent to K with the spectral condition number estimate

$$(\underline{\gamma}\,/\,\overline{\gamma}) \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2 \;\le\; \kappa(C^{-1}K) \;\le\; (\overline{\gamma}\,/\,\underline{\gamma}) \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2 \tag{2}$$

Here $\mu := \varrho\,(S_C^{-1}T_C)$ is the spectral radius of $S_C^{-1}T_C$, and $\underline{\gamma} := \min\{\underline{\gamma}_C, \underline{\gamma}_I\}$ and $\overline{\gamma} := \max\{\overline{\gamma}_C, \overline{\gamma}_I\}$.

In our examples the three components C_I, C_C and B_I of this preconditioner C are chosen as follows:

- C_I is defined by a suitable symmetric multigrid method yielding h-independent constants $\underline{\gamma}_I$ and $\overline{\gamma}_I = 1$ (cf. [17]).
- For C_C the classical Dryja preconditioner (cf. [5]) is used in the model problem 1 giving h-independent constants $\underline{\gamma}_C$ and $\overline{\gamma}_C$ as well. In example 2, the BPS preconditioner [2] is applied resulting in a relatively slowly growing ratio $\overline{\gamma}_C/\underline{\gamma}_C = O(\ln^2 h^{-1})$ as $h \to 0$.
- B_I is implicitly defined by the iteration operator $\overline{M}_I$ of some iteration method, i.e.

  $$B_I = K_I \left(I_I - \overline{M}_I\right)^{-1} \quad . \tag{3}$$

  For model problem 1 and for several operators $\overline{M}_I$ the behaviour of the spectral radius $\mu = \varrho(S_C^{-1}T_C)$ as $h \to 0$ is as follows:

  - The two-grid operator (i.e. multigrid with two grids) led to $\mu = O(h^{-1})$ on numerical examples (cf. Haase [10]).
  - The application of one multigrid step resulted in a growing $\mu = \varrho\,(S_C^{-1}T_C)$ as $h \to 0$ on numerical tests [10]. Theoretically the growth is at most $O(h^{-1})$ [12]. However, $O(\ln h^{-1})$ multigrid steps ensure a bounded μ.

- Haase, Langer, Meyer, and Nepomnyaschikh [14] construct the initial guess of the multigrid iteration by means of a hierarchical extension procedure. This gives $\mu = \varrho(S_C^{-1}T_C) = O(\ln h^{-1})$, or equivalently, $O(\ln \ln h^{-1})$ multigrid steps imply a bounded μ.

  Haase [11] improved the extension technique and showed that $O(\ln \ln h^{-1})$ smoothing steps yield a bounded μ.

  Recently Nepomnyaschikh [21] proposed a BPX-like extension operator resulting in a bounded μ without multigrid improvement.

- Haase [10] applied suitable frequency filter methods yielding an apparently bounded μ in numerical experiments.

These theoretical and numerical results indicate that the spectral condition number $\kappa(C^{-1}K)$ depends heavily on $\mu = \varrho(S_C^{-1}T_C)$ and thus on $T_C = K_{CI}(K_I^{-1} - B_I^{-T})K_I(K_I^{-1} - B_I^{-1})K_{IC}$. Hence the improvement of $\kappa(C^{-1}K)$ has to be achieved via the basis transformation B_I.

To our knowledge the use of the full multigrid technique for the definition of the basis transformation B_I has not been investigated theoretically. The analysis of this method is a major aim of this paper. We try to find out as to whether the h-dependence of $\mu = \varrho(S_C^{-1}T_C)$, as it occurs by applying only one multigrid step, can be overcome.

4 Model problem analysis

As outlined above we intend to analyse the full *multigrid* method for defining the basis transformation B_I. Due to the very complex matter however we have to restrict the analysis to the full *two-grid* operator and the model problem 1 instead.

In this section the basic formulae to evaluate $\mu = \varrho(S_C^{-1}T_C)$ are derived first, followed by the analysis of the full two-grid operator. For the special case where the relaxation parameter is chosen to be $\omega = 1.0$ a proof of the growth of μ is presented in the remainder of this section.

4.1 Basic formulae

We consider the model problem 1 (cf. Figure 1). Following [10] the basis transformation matrix B_I as defined in (3) leads to

$$\mu = \varrho(S_C^{-1}T_C) = \sup_{\underline{u}_C \in \mathrm{R}^{\mathrm{N_C}} \setminus \{\mathrm{O}\}} \frac{(T_C \underline{u}_C\,,\, \underline{u}_C)}{(S_C \underline{u}_C\,,\, \underline{u}_C)} = \sup_{\underline{u}_C \in \mathrm{R}^{\mathrm{N_C}} \setminus \{\mathrm{O}\}} \frac{\|\overline{M}_I K_I^{-1} K_{IC}\, \underline{u}_C\|^2_{K_I}}{\|\underline{u}_C\|^2_{S_C}}$$

$$= \sup_{\underline{u}_C \in \mathrm{R}^{\mathrm{N_C}} \setminus \{\mathrm{O}\}} \frac{\sum_{i=1}^{p} \|\overline{M}_{I,i} K_{I,i}^{-1} K_{IC,i}\, \underline{u}_C\|^2_{K_{I,i}}}{\|\underline{u}_C\|^2_{S_C}}$$

because of the block structure of $\overline{M}_I = \mathrm{diag}\{\overline{M}_{I,i}\}_{i=1,p}$, of $K_{IC} = K_{CI}^T = [K_{IC,1} \ldots, K_{IC,p}]$, and of $K_I = \mathrm{diag}\{K_{I,i}\}_{i=1,p}$ (with $p = 2$ subdomains). Introducing

$$\delta_i := \sup_{\underline{u}_C \in \mathrm{R}^{\mathrm{N_C}} \setminus \{\mathrm{O}\}} \frac{\|\overline{M}_{I,i} K_{I,i}^{-1} K_{IC,i}\, \underline{u}_C\|^2_{K_{I,i}}}{\|\underline{u}_C\|^2_{S_C}} \tag{4}$$

gives $\mu \leq \sum_{i=1}^{p} \delta_i$. The terms of equation (4) will now be evaluated by means of the Fourier analysis which utilizes the basic ideas of Stüben and Trottenberg [16].

Let the indices k and i correspond to the x-directions and l and j to the y-direction. Let $n = 1/h$ be the (even) number of intervals on $[0, 1]$. The eigenvectors and the eigenvalues of the

matrix K_I are (cf. [22])

$$\mu_{k,l} \;=\; \{\, \mu_k(i) \cdot \mu_l(j) \,\}_{i,j=0\ldots n} \qquad \text{and} \qquad \lambda_{k,l} \;=\; \lambda_k + \lambda_l \qquad k,l = 1 \ldots n-1$$
$$\text{with} \qquad \mu_k(i) \;=\; \sqrt{2h}\, \sin(k\pi i h) \qquad \text{and} \qquad \lambda_k \;=\; 4\sin^2(k\pi h/2) \qquad .$$

Let the Fourier expansion of $\underline{u}_C$ (on the coupling boundary Γ_C) be

$$u_C(j) \;=\; \sum_{l=1}^{n-1} \alpha_l\, \mu_l(j) \qquad \text{with} \qquad \alpha_l \;=\; \left(\underline{u}_C\,,\, \underline{\mu}_l\right) \;=\; \sum_{j=1}^{n-1} u_C(j)\mu_l(j) \qquad .$$

Then the denominator of (4) can be approximated as

$$\|\underline{u}_C\|^2_{S_C} \;=\; (S_C\underline{u}_C\,,\, \underline{u}_C) \;\approx\; \sum_{l=1}^{n-1} \sqrt{\lambda_l^2 + 4\lambda_l}\;\, \alpha_l^2 \tag{5}$$

which is asymptotically exact as $h \to 0$ (cf. Golub [9] and Haase [10]).

The numerator $\|\overline{M}_{I,i}K_{I,i}^{-1}K_{IC,i}\,\underline{u}_C\|^2_{K_{I,i}}$ of (4) has been derived in Haase [10] as follows:

$$\underline{y}_I \;:=\; K_{IC}\underline{u}_C \;=\; \sum_{l=1}^{n-1}\sum_{k=1}^{n-1} \left(\, -\mu_k(n-1)\;\alpha_l \,\right)\; \mu_{k,l}$$

$$\underline{z}_I \;:=\; K_I^{-1}K_{IC}\underline{u}_C \;=\; K_I^{-1}\underline{y}_I \;=\; \sum_{l=1}^{n-1}\sum_{k=1}^{n-1} \left(\frac{-\mu_k(n-1)}{\lambda_{k,l}}\;\alpha_l \right)\; \mu_{k,l} \tag{6}$$

$$\|\overline{M}_I K_I^{-1}K_{IC}\;\underline{u}_C\|^2_{K_I} \;=\; \|\overline{M}_I\underline{z}_I\|^2_{K_I} \;=\; \left(K_I\overline{M}_I\underline{z}_I\,,\, \overline{M}_I\underline{z}_I\right) \qquad . \tag{7}$$

With these expansions (5) and (7) the fraction δ_i and thus the desired spectral radius $\mu = \varrho\,(S_C^{-1}T_C)$ will be evaluated. Equation (7) is of special interest since it contains the iteration operator $\overline{M}_I$ which describes the basis transformation B_I (cf. (3)). The next subsection is devoted to the application of the full two-grid operator for the definition of $\overline{M}_I$.

4.2 The full two-grid operator

Let $\hat{M}_I$ and M_I denote the two-grid operator and the full two-grid operator, respectively. We will expand M_I and then obtain an exact formula for $\|M_I\underline{z}_I\|^2_{K_I}$ and eventually for μ.

We start by introducing the abbreviations

$$k' \;=\; n-k \quad , \quad s_k \;=\; \sin^2(k\pi h/2)\;,\; c_k \;=\; \cos^2(k\pi h/2)$$

which yield obviously $s_{k'} = c_k$ and $c_{k'} = s_k$. Let $[k,l] := \{(k,l),(k',l),(k,l'),(k',l')\}$ be an index quadruplet. Let $\underline{x}_{[k,l]} := [x_{k,l}, x_{k',l}, x_{k,l'}, x_{k',l'}]^T$ be that 4-dimensional subvector of a vector $\underline{x} = \{x_{k,l}\}_{k,l=\overline{1,n-1}}$ that corresponds to $[k,l]$. Analogously let $X_{[k,l]}$ denote the 4×4 submatrix of a (diagonal or block-diagonal) matrix X whose entries relate to $[k,l]$. Let $[l] := \{l,l'\}$ be an index pair. Then $\sum_{[l]}$ and $\sum_{[k,l]}$ shall denote the summation over all index pairs and index quadruplets, respectively.

Let us now derive the expansion of $\hat{M}_I$ and M_I. The two-grid operator $\hat{M}_I$ can be written as

$$\hat{M}_I \;=\; S_h\; C_h\; S_h \;=\; \left(I_h - \frac{\omega}{4}K_h\right)\cdot\left(I_h - I_H^h K_H^{-1} I_h^H K_h\right)\cdot\left(I_h - \frac{\omega}{4}K_h\right) \qquad .$$

The indices h and H are related to the fine and coarse grid, respectively. We apply one ω-Jacobi iteration for pre- and post-smoothing. The following notation is used.

S_h	–	pre- and post-smoothing operator with smoothing parameter ω
C_h	–	coarse grid correction operator
I_h	–	identity matrix
K_h, K_H	–	stiffness matrix K of the fine/coarse grid
I_H^h	–	bilinear interpolation operator from the coarse onto the fine grid
I_h^H	–	restriction operator from the fine onto the coarse grid

The full two-grid technique means that the coarse grid system $K_H\, u_H = r_H$ is solved first before taking the interpolant $u_{k+\frac{1}{2}} = I_H^h\, u_H$ as the initial guess of the subsequent two-grid iteration. Thus we have

$$\begin{aligned} u - u_{k+1} &= \hat{M}_I\,(u - u_{k+\frac{1}{2}}) = \hat{M}_I\,(u - I_H^h\, u_H) = \hat{M}_I\,(u - I_H^h\, K_H^{-1}\, r_H) \\ &= \hat{M}_I\,(u - I_H^h\, K_H^{-1}\, I_h^H\, r_h) = \hat{M}_I\,(u - I_H^h\, K_H^{-1}\, I_h^H\,(f - K_h\, u_k)) \\ &= \hat{M}_I\,(u - I_H^h\, K_H^{-1}\, I_h^H\, K_h\,(u - u_k)) = \hat{M}_I\,(I - I_H^h\, K_H^{-1}\, I_h^H\, K_h)(u - u_k) \end{aligned}$$

(since we start with $u_k = 0$). Therefore the error transition operator of the full two-grid operator can be expressed as

$$M_I = \hat{M}_I\,(I - I_H^h\, K_H^{-1}\, I_h^H\, K_h) = S_h\, C_h\, S_h \cdot C_h \qquad .$$

This operator M_I will be expanded with respect to the orthonormal basis

$$Q := \{\mu_{k,l}\}_{k,l=\overline{1,n-1}}$$

of the eigenvectors of K_I. Let the eigenvectors of an index quadruplet $[k,l]$ be placed in adjacent columns. The analysis of the appropriate operators results in

$$\left.\begin{array}{rrcl} & K_I \cdot Q &=& Q \cdot \Lambda \\ & S_h \cdot Q &=& Q \cdot \Theta \\ & C_h \cdot Q &=& Q \cdot C \\ \text{and thus} & M_I \cdot Q &=& S_h C_h S_h \cdot C_h \cdot Q = Q \cdot \Theta\, C\, \Theta\, C \end{array}\right\} \qquad (8)$$

The matrix Λ occurring in (8) has the form

$$\Lambda = \mathrm{diag}\{\lambda_{k,l}\}_{k,l=\overline{1,n-1}} \;=\; 4 \cdot \mathrm{diag}\{s_k + s_l\}_{k,l=\overline{1,n-1}} \qquad .$$

The matrix Θ represents the ω-Jacobi smoothing and thus expands to

$$\Theta = \mathrm{diag}\left\{\left(1 - \frac{\omega}{4}\lambda_{k,l}\right)\right\} \qquad .$$

The coarse grid correction matrix C has been derived in [16] and [10] to be

$$\begin{aligned} C &= \mathrm{blockdiag}\{C_{[k,l]}\} \\ \text{with} \qquad C_{[k,l]} &= I - R \cdot c\, c^T \Lambda_{[k,l]} \qquad (9) \\ \text{and} \qquad I &= \text{the } 4 \times 4 \text{ identity matrix,} \\ R &:= \frac{1}{4(s_k c_k + s_l c_l)}, \qquad (10) \\ c &:= [c_k c_l,\, -s_k c_l,\, -c_k s_l,\, s_k s_l]^T \qquad . \qquad (11) \end{aligned}$$

Here $\Lambda_{[k,l]}$ is the corresponding submatrix of Λ, i.e

$$\Lambda_{[k,l]} = 4 \cdot \mathrm{diag}\{s_k + s_l\,,\; c_k + s_l\,,\; s_k + c_l\,,\; c_k + c_l\} \qquad .$$

Note that for $k = k'$ and/or $l = l'$ the matrix $C_{[k,l]}$ is reduced to the 2×2 or 1×1 identity matrix. Additionally the matrix $\Theta C \Theta C$ is block-diagonal and thus the eigen subspace related to the quadruplet $[k,l]$ remains unchanged under the application of the full two-grid operator M_I.

Let us repeat the Fourier expansion (6) of $\underline{z}_I$:

$$\begin{aligned} \underline{z}_I = \sum_{l=1}^{n-1}\sum_{k=1}^{n-1} \left(\frac{-\mu_k(n-1)}{\lambda_{k,l}}\,\alpha_l \right) \mu_{k,l} &= \sum_{l=1}^{n-1}\sum_{k=1}^{n-1} \beta_{k,l}\mu_{k,l} = Q \cdot \underline{\beta} \\ \text{with} \qquad \beta_{k,l} &:= \frac{-\mu_k(n-1)}{\lambda_{k,l}}\,\alpha_l \qquad . \end{aligned} \tag{12}$$

Utilizing the matrix relations (8) we conclude

$$\begin{aligned} M_I \underline{z}_I &= M_I \cdot Q\underline{\beta} = Q \cdot \Theta C \Theta C \cdot \underline{\beta} \\ K_I M_I \underline{z}_I &= K_I \cdot Q \cdot \Theta C \Theta C \cdot \underline{\beta} = Q \cdot \Lambda \cdot \Theta C \Theta C \cdot \underline{\beta} \\ \Longrightarrow \qquad (K_I M_I \underline{z}_I, M_I \underline{z}_I) &= (Q \cdot \Lambda \cdot \Theta C \Theta C \cdot \underline{\beta}, Q \cdot \Theta C \Theta C \cdot \underline{\beta}) \\ &= (C^T \Theta C^T \Theta \cdot \Lambda \cdot \Theta C \Theta C \underline{\beta}, \underline{\beta}) \end{aligned}$$

since Q is an orthonormal basis. We introduce the new block-diagonal matrix

$$\begin{aligned} F &:= C^T \Theta C^T \Theta \cdot \Lambda \cdot \Theta C \Theta C = \text{blockdiag}\{F_{[k,l]}\} \\ \text{with} \qquad F_{[k,l]} &= C^T_{[k,l]}\,\Theta_{[k,l]}\,C^T_{[k,l]}\,\Theta_{[k,l]} \cdot \Lambda_{[k,l]} \cdot \Theta_{[k,l]}\,C_{[k,l]}\,\Theta_{[k,l]}\,C_{[k,l]} \in \mathbb{R}^{4\times 4} \end{aligned} \tag{13}$$

and obtain the intermediate result

$$\|M_I \underline{z}_I\|^2_{K_I} = (F\underline{\beta}, \underline{\beta}) = \sum_{[k,l]} (F_{[k,l]}\,\underline{\beta}_{[k,l]}, \underline{\beta}_{[k,l]}) \qquad .$$

The substitution (12) of the $\beta_{k,l}$ by the α_l can be written as

$$\begin{aligned} \beta_{k,l} &= \frac{-\mu_k(n-1)}{\lambda_{k,l}}\,\alpha_l \\ \text{or} \qquad \underline{\beta}_{[k,l]} &= \Gamma_{[k,l]} \cdot \underline{\alpha}_{[l]} \\ \text{with} \qquad \Gamma_{[k,l]} &= \begin{bmatrix} -\dfrac{\mu_k(n-1)}{\lambda_{k,l}} & 0 \\ -\dfrac{\mu_{k'}(n-1)}{\lambda_{k',l}} & 0 \\ 0 & -\dfrac{\mu_k(n-1)}{\lambda_{k,l'}} \\ 0 & -\dfrac{\mu_{k'}(n-1)}{\lambda_{k',l'}} \end{bmatrix} \in \mathbb{R}^{4\times 2} \end{aligned} \tag{14}$$

and the definition $\underline{\alpha}_{[l]} := [\alpha_l, \alpha_{l'}]^T$. The application of this substitution results in

$$\begin{aligned} \|M_I \underline{z}_I\|^2_{K_I} &= (F\underline{\beta}, \underline{\beta}) = \sum_{[k,l]} (F_{[k,l]}\,\underline{\beta}_{[k,l]}, \underline{\beta}_{[k,l]}) = \sum_{[k,l]} (F_{[k,l]}\Gamma_{[k,l]}\underline{\alpha}_{[l]}, \Gamma_{[k,l]}\underline{\alpha}_{[l]}) \\ &= \sum_{[k,l]} (\Gamma^T_{[k,l]} F_{[k,l]} \Gamma_{[k,l]}\underline{\alpha}_{[l]}, \underline{\alpha}_{[l]}) = \sum_{[k,l]} (G_{[k,l]}\underline{\alpha}_{[l]}, \underline{\alpha}_{[l]}) = \sum_{[l]} \left(\sum_{[k]} G_{[k,l]}\underline{\alpha}_{[l]}, \underline{\alpha}_{[l]} \right) \\ &= \sum_{[l]} \left(H_{[l]}\underline{\alpha}_{[l]}, \underline{\alpha}_{[l]} \right) = (H\underline{\alpha}, \underline{\alpha}) \end{aligned}$$

with the new matrices

$$\begin{aligned} G_{[k,l]} &:= \Gamma_{[k,l]}^T F_{[k,l]} \Gamma_{[k,l]} \quad \in \quad \mathbb{R}^{2\times 2} \\ H_{[l]} &:= \sum_{[k]} G_{[k,l]} \quad \in \quad \mathbb{R}^{2\times 2} \\ \text{and} \qquad H &:= \text{blockdiag}\left\{H_{[l]}\right\} \qquad . \end{aligned} \tag{15}$$

Thus the numerator of δ_i in (4) is evaluated.

The denominator of δ_i has been approximated in (5) as

$$\|\underline{u}_C\|_{S_C}^2 \;=\; (S_C \underline{u}_C \,,\, \underline{u}_C) \;\approx\; \sum_{l=1}^{n-1} \sqrt{\lambda_l^2 + 4\lambda_l}\;\alpha_l^2 \;=\; \sum_{[l]} \left(D_{[l]} \underline{\alpha}_{[l]}, \underline{\alpha}_{[l]} \right) \;=\; (D\underline{\alpha}, \underline{\alpha}) \qquad .$$

Here $D_{[l]}$ and D denote the positive definite diagonal matrices

$$\begin{aligned} D_{[l]} &= \text{diag}\left\{ \sqrt{\lambda_l^2 + 4\lambda_l} \,,\, \sqrt{\lambda_{l'}^2 + 4\lambda_{l'}} \right\} \\ \text{and} \qquad D &= \text{blockdiag}\left\{D_{[l]}\right\} \qquad . \end{aligned} \tag{16}$$

Finally δ_i can be expressed as

$$\delta_i \;=\; \sup_{\underline{u}_C \in \mathrm{R}^{\mathrm{N_C}} \setminus \{\mathrm{O}\}} \frac{\|M_I \underline{z}_I\|_{K_I}^2}{\|\underline{u}_C\|_{S_C}^2} = \max_{\underline{\alpha}} \frac{(H\underline{\alpha}, \underline{\alpha})}{(D\underline{\alpha}, \underline{\alpha})} \;=\; \varrho\left(D^{-1}H\right) = \max_{[l]} \;\varrho\left(D_{[l]}^{-1} H_{[l]}\right) \qquad . \tag{17}$$

since $D^{-1}H$ has a 2×2 block-diagonal form.

This last equation has been too difficult to analyse. It allows, however, an exact numerical evaluation of δ_i and thus of $\mu = \varrho\left(S_C^{-1} T_C\right)$. Table 1 comprises the the computed values of δ_i for decreasing h and for three smoothing parameters ω.

$n = 1/h$	$\omega = 0.5$	$\omega = 1.0$	$\omega = 1.5$
4	3.6368 E-2	2.9302 E-3	6.6638 E-2
8	1.2118 E-1	1.5418 E-2	8.9133 E-2
16	2.8438 E-1	3.9860 E-2	9.5985 E-2
32	5.9469 E-1	8.5136 E-2	9.7809 E-2
64	1.2034	1.7312 E-1	9.8276 E-2
128	2.4141	3.4767 E-1	9.8395 E-2
256	4.8319	6.9606 E-1	9.8428 E-2
512	9.6656	1.3925	1.9639 E-1
1024	19.332	2.7852	3.9288 E-1

Table 1: Calculation of δ_i for the full two-grid operator

Note that the maximum in (17) always occurs at $l_{max} = 1$. The results presented suggest the following

Conjecture 1 *Consider the model problem 1 (cf. Figure 1). Suppose the full two-grid operator with bilinear interpolation and ω-Jacobi smoothing is used as M_I. Then the spectral radius $\mu = \varrho\left(S_C^{-1} T_C\right)$ apparently grows like $O(h^{-1})$ as $h \to 0$.*

Remark 1 The growth of $\mu = O(h^{-1})$ is obvious for $\omega = 0.5$ and $\omega = 1.0$. In order to understand the different behaviour for $\omega = 1.5$ we have to look at equation (17). The two diagonal entries of $D_{[1]}^{-1} H_{[1]}$ behave differently (in the numerical test). The right lower entry (related to $\underline{u}_C = \mu_{n-1}$) seems to be bounded and dominates the spectral radius if $n \leq 256$. The left upper entry however (related to $\underline{u}_C = \mu_1$) grows like $O(h^{-1})$ and takes over for $n > 256$.

Remark 2 The dependence of δ_i on the smoothing parameter ω is shown in Figure 3. Four different mesh sizes $h = 1/n$ are considered.

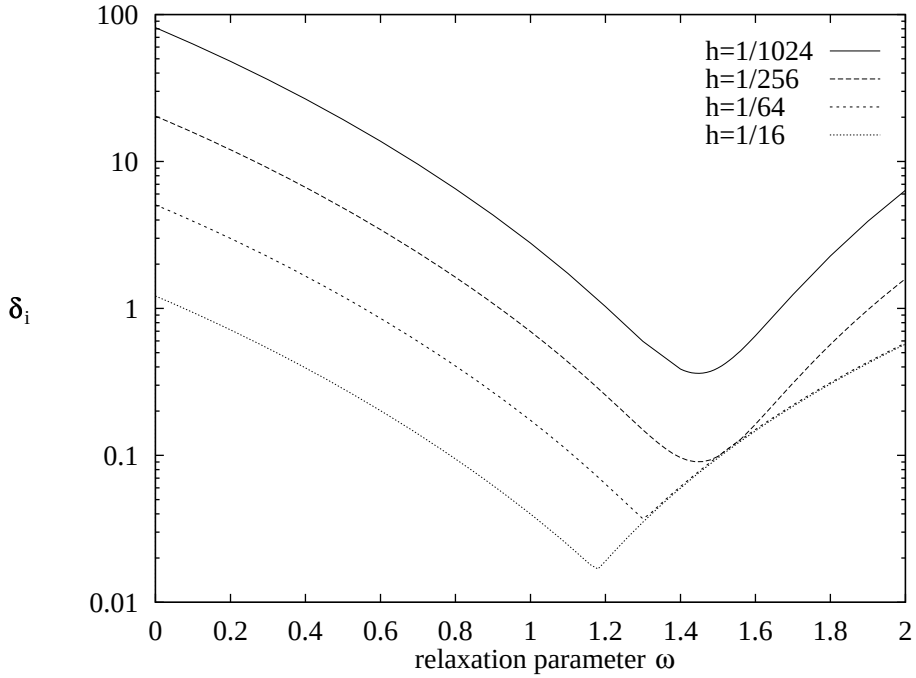

Figure 3: Dependence of δ_i on ω for different h

Remark 3 For the multigrid method discussed here bilinear interpolation and the transposed restriction is assumed. The common FEM interpolation is not investigated since then (to our knowledge) no useful Fourier analysis of the coarse grid correction matrix C is possible.

4.3 Analysis of a special full two-grid operator

For the special case where the smoothing parameter is chosen to be $\omega = 1$ a proof of the growth of μ can be established.

Theorem 2 *Consider the model problem 1 and the full two-grid operator as M_I. Suppose that bilinear interpolation and ω-Jacobi smoothing with $\omega = 1$ is used. Then the behaviour of the spectral radius is $\mu = \varrho\,(S_C^{-1} T_C) \geq O(h^{-1})$ as $h \to 0$.*

Proof: For convenience we assume that $n \geq 8$ is a multiple of 4.

Let the upper index (in parentheses) of a matrix or a vector refer to the corresponding entry. With this notation we have

$$\delta_i \;=\; \max_{[l]} \varrho\left(D_{[l]}^{-1} H_{[l]}\right) \geq \varrho\left(D_{[1]}^{-1} H_{[1]}\right) \geq \left(D_{[1]}^{(1,1)}\right)^{-1} \cdot H_{[1]}^{(1,1)}$$

$$\approx \quad \frac{1}{4\sqrt{s_1^2+s_1}} \cdot \sum_{[k]} G_{[k,1]}^{(1,1)} \;\geq\; \frac{1}{4\sqrt{s_1^2+s_1}} \cdot \sum_{n/4}^{n/2-1} G_{[k,1]}^{(1,1)} \qquad .$$

since $H_{[1]}$ is symmetric. Additionally $\Lambda_{[k,1]}$ and thus $G_{[k,1]}$ (and $H_{[1]}$) are positively semidefinit, and therefore $G_{[k,1]}^{(1,1)} \geq 0$.

We restrict k to values $n/4 \leq k < n/2$ and aim for a bound of $G_{[k,1]}^{(1,1)}$ for such k. For simplicity we will omit the index quadruplet $[k,1]$ of the matrices $G_{[k,1]}, \Lambda_{[k,1]}, \Theta_{[k,1]}$, $C_{[k,1]}$ and $\Gamma_{[k,1]}$. From (15) we obtain

$$\begin{aligned} G_{[k,1]} = G &= \Gamma^T \cdot C^T\,\Theta\,C^T\,\Theta \cdot \Lambda \cdot \Theta\,C\,\Theta\,C \cdot \Gamma \\ \text{and} \qquad G^{(1,1)} &= e_1^T \cdot G \cdot e_1 \end{aligned}$$

with $e_1 := (1,0)^T$ being the first unitary vector. Utilizing the trigonometric identities we conclude

$$\begin{aligned} \Gamma \cdot e_1 &= \left(-\frac{\mu_k(n-1)}{\lambda_{k,1}}\,,\, -\frac{\mu_{k'}(n-1)}{\lambda_{k',l}}\,,\, 0\,,\, 0 \right)^T \\ &= \sqrt{2h} \cdot (-1)^k \cdot \sin(k\pi h) \cdot \left(\frac{1}{\lambda_{k,1}}\,,\, \frac{1}{\lambda_{k',1}}\,,\, 0\,,\, 0 \right)^T \\ &=: \eta \cdot d \end{aligned}$$

with the new notation

$$\eta^2 := 2h \cdot \sin^2(k\pi h) = 8h \cdot s_k c_k \qquad \text{and} \qquad d := \left(\frac{1}{\lambda_{k,1}}\,,\, \frac{1}{\lambda_{k',1}}\,,\, 0\,,\, 0 \right)^T \qquad .$$

This leads to

$$\begin{aligned} G^{(1,1)} &= \eta^2 \cdot d^T\,C^T \cdot \Theta\,C^T\,\Theta \cdot \Lambda \cdot \Theta\,C\,\Theta \cdot C\,d \\ &= \eta^2 \cdot b^T \cdot A^T \cdot \Lambda \cdot A \cdot b \end{aligned}$$

$$\text{with} \qquad A := \Theta\,C\,\Theta \in \mathbb{R}^{4\times 4} \qquad \text{and} \qquad b := C\,d \in \mathbb{R}^4 \qquad .$$

All entries of the diagonal matrix Λ are positive. This implies

$$\begin{aligned} G^{(1,1)} &= \eta^2 \cdot \sum_{j=1}^{4} \left(b^T\,A^T\right)^{(j)} \cdot \Lambda^{(j)} \cdot (A\,b)^{(j)} \\ &\geq \eta^2 \cdot \left(b^T\,A^T\right)^{(1)} \cdot \Lambda^{(1)} \cdot (A\,b)^{(1)} = \eta^2 \cdot \lambda_{k,1} \cdot \left((A\,b)^{(1)}\right)^2 \qquad . \end{aligned}$$

The first entry of $A \cdot b$ can be written as

$$(A\,b)^{(1)} = (1,0,0,0) \cdot A \cdot b = e_1^T\,A \cdot b$$

with $e_1 := (1,0,0,0)^T$ being the first unitary vector. We will now investigate the vectors $e_1^T\,A$ and b.

Let us start with $e_1^T\,A = e_1^T \cdot \Theta\,C\,\Theta =: (a_1\,,\,a_2\,,\,a_3\,,\,a_4)$. We perform one ω-Jacobi smoothing step with $\omega = 1$. The corresponding matrix Θ becomes

$$\Theta = I - \frac{\omega}{4}\Lambda = \text{diag}\,\{c_k - s_1\,,\, s_k - s_1\,,\, c_k - c_1\,,\, s_k - c_1\} \qquad .$$

In (9) the coarse grid correction matrix $C = I - R \cdot c\, c^T \Lambda$ has been derived. A cumbersome but straight-forward calculation results in

$$e_1^T \cdot \Theta\, C\, \Theta \;=\; \begin{pmatrix} (c_k - s_1)^2 \;-\; \frac{(c_k - s_1)\cdot c_k c_1}{s_k c_k + s_1 c_1} c_k c_1 \cdot (s_k + s_1) \cdot (c_k - s_1) \\ \frac{(c_k - s_1)\cdot c_k c_1}{s_k c_k + s_1 c_1} s_k c_1 \cdot (c_k + s_1) \cdot (s_k - s_1) \\ -\frac{(c_k - s_1)\cdot c_k c_1}{s_k c_k + s_1 c_1} c_k s_1 \cdot (s_k + c_1) \cdot (c_1 - c_k) \\ \frac{(c_k - s_1)\cdot c_k c_1}{s_k c_k + s_1 c_1} s_k s_1 \cdot (c_k + c_1) \cdot (c_1 - s_k) \end{pmatrix}^T$$

Using $0 < s_1 \le s_k < 1/2 < c_k \le c_1 < 1$ we obtain the inequalities

$$\begin{aligned} c_k^2 c_1^2 \frac{s_k + s_1}{s_k c_k + s_1 c_1} &\le c_k c_1^2 \frac{s_k c_k + s_1 c_k}{s_k c_k + s_1 c_1} \le c_k \\ \text{and} \qquad a_1 &= (c_k - s_1)^2 \cdot \left(1 - c_k^2 c_1^2 \frac{s_k + s_1}{s_k c_k + s_1 c_1}\right) \\ &\ge (c_k - s_1)^2 \cdot (1 - c_k) \;=\; s_k \cdot (c_k - s_1)^2 \\ &\ge s_k \cdot (c_{\frac{n}{2}-1} - s_1)^2 \;=\; s_k \cdot \left(\cos^2((\frac{n}{2} - 1)\pi h/2) - \sin^2(\pi h/2)\right)^2 \\ &= s_k \cdot \frac{1}{2}\cos^2\left(\frac{\pi}{4} - \pi h\right) \ge \frac{1}{4} s_k \qquad . \end{aligned}$$

Analogously the following bounds of the other entries $a_2 \ldots a_4$ are derived:

$$\begin{aligned} \tfrac{1}{4}\, s_k &\le a_1 \\ \tfrac{1}{4}\, s_k &\le a_2 \qquad\qquad \text{for } k \ge n/4 \\ -s_1 &\le a_3 \le 0 \\ 0 &\le a_4 \le 2 s_1 \end{aligned}$$

The vector $b = C\, d =: (b_1\,,\, b_2\,,\, b_3\,,\, b_4)^T$ is dealt with in a similar manner. From the expansion of C and d we conclude

$$b \;=\; \frac{1}{4} \cdot \begin{pmatrix} \frac{1}{s_k + s_1} \;-\; \frac{c_1 \cdot (c_k - s_k)}{s_k c_k + s_1 c_1} \cdot c_k c_1 \\ \frac{1}{c_k + s_1} \;+\; \frac{c_1 \cdot (c_k - s_k)}{s_k c_k + s_1 c_1} \cdot s_k c_1 \\ \frac{c_1 \cdot (c_k - s_k)}{s_k c_k + s_1 c_1} \cdot c_k s_1 \\ -\frac{c_1 \cdot (c_k - s_k)}{s_k c_k + s_1 c_1} \cdot s_k s_1 \end{pmatrix}$$

As above, we similarly obtain

$$\begin{aligned} \frac{1}{4} &\le b_1 \\ \frac{1}{4} &\le b_2 \\ 0 &\le b_3 \le \frac{1}{8} \\ -\frac{1}{64} &\le b_4 \le 0 \end{aligned}$$

These estimates for a_j and b_j result in

$$\begin{aligned} (A\,b)^{(1)} &= e_1^T\,A\cdot b = \sum_{j=1}^{4} a_j\cdot b_j \;\geq\; \frac{1}{8}\,s_k - \frac{5}{32}s_1 \\ (A\,b)^{(1)} &\geq \frac{1}{16}\,s_k \qquad\qquad \text{if } k\geq n/4\,,\; n\geq 8 \\ \text{and}\qquad G^{(1,1)} &\geq \eta^2\cdot\lambda_{k,1}\cdot\Big((A\,b)^{(1)}\Big)^2 \;\geq\; 8h\cdot s_k c_k\cdot 4(s_k+s_1)\cdot\frac{1}{256}s_k^2 \\ &\geq \frac{1}{4}\,h\cdot s_k^4 \qquad\qquad \text{if } k\geq n/4\,,\; n\geq 8 \end{aligned}$$

Now the matrix entry $H_{[1]}^{(1,1)}$ can be bounded according to

$$\begin{aligned} H_{[1]}^{(1,1)} = H^{(1,1)} &= \sum_{[k]} G^{(1,1)} \;>\; \sum_{k=n/4}^{n/2-1} G^{(1,1)} \;>\; \frac{1}{4}\sum_{k=n/4}^{n/2-1} h\cdot\big(\sin^2(k\pi h/2)\big)^4 \\ &> \frac{1}{4}\int_{\frac{1}{4}-\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \sin^8(\frac{\pi}{2}x) \;>\; 0.0001 \qquad \text{for } n\geq 8 \qquad . \end{aligned}$$

Since $s_1 = \sin^2(\pi * h/2) = O(h^2)$ we conclude

$$\begin{aligned} 4\sqrt{s_1^2+s_1} &= O(h) \\ \text{and}\qquad \delta_i \;\geq\; \frac{1}{4\sqrt{s_1^2+s_1}}\cdot H_{[1]}^{(1,1)} &= O(h^{-1}) \qquad . \end{aligned}$$

Thus the desired spectral radius $\mu = 2\delta_i$ grows (at least) like $O(h^{-1})$ as $h\to 0$. ■

5 Numerical experiments

We apply the parallel preconditioned cg algorithm (ppcg) to model problem 1 and 2, respectively [12, 13]. We utilize the preconditioner C of (1)

$$C = \begin{pmatrix} I_C & K_{CI}B_I^{-T} \\ O & I_I \end{pmatrix}\begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix}\begin{pmatrix} I_C & O \\ B_I^{-1}K_{IC} & I_I \end{pmatrix} \qquad ,$$

where B_I is defined either by multigrid or full multigrid and C_I and C_C are given below.

After investigating $\mu = \varrho(S_C^{-1}T_C)$ in section 4 we now focus on the number of ppcg iterations that are required to reach a given relative accuracy ($\varepsilon = 10^{-6}$). This number of iterations and μ are linked via the spectral condition number $\kappa(C^{-1}K)$ (cf. section 3 and equation (2) for the theoretical background).

Our experiments are twofold. The first part of this section is devoted solely to the verification of our model problem analysis. The remaining part contains the actual comparison of the ppcg iteration numbers for both the multigrid and full multigrid basis transformation.

Finally let MG and FMG denote multigrid and full multigrid.

5.1 Verification of the model problem analysis

Here we only verify the model problem analysis of section 4 by applying the ppcg algorithm to model problem 1. The three components of the preconditioner C of (1) are as follows:

- The basis transformation B_I is defined by one full two-grid step. Three different parameters of the ω-Jacobi smoother are tested.
- The well-known Dryja preconditioner serves as C_C.
- We set $C_I := K_I$.

Let h be the coarse grid mesh size. The last two components ensure $\underline{\gamma} \,/\, \overline{\gamma} = O(1)$. B_I has been investigated in section 4, and the theoretical results are comprised in Table 1, in conjecture 1, in Figure 3, and in theorem 2. They strongly suggest $\mu = \varrho(S_C^{-1}T_C) = O(h^{-1})$ as $h \to 0$. According to (2) we expect the number of cg-iterations to be $O(\sqrt{\kappa(C^{-1}K)} = O(\mu^{1/2})$.

The computed values of $\mu = 2\delta_i$ are contained in Table 1. Hence for $\omega = 0.5$ and $\omega = 1.0$ the number of iterations should be $O(h^{-1/2})$ as $h \to 0$.. For $\omega = 1.5$ we expect bounded iteration numbers for $h \geq 1/256$; for smaller h the iteration numbers should grow as $O(h^{-1/2})$. Table 2 presents the numerical results.

$n = 1/h$	4	8	16	32	64	128	256	512
$\omega = 0.5$	5	7	8	10	12	18	23	37
$\omega = 1.0$	5	6	6	7	7	9	10	13
$\omega = 1.5$	5	7	7	8	7	7	7	8

Table 2: Number of cg-iterations for the full two-grid operator

For a fixed ω the iteration numbers show the anticipated behaviour and thus verify the model problem analysis.

5.2 Comparison of multigrid versus full multigrid

These practical computations concentrate on the number of pccg iterations. The three components B_I, C_I and C_C of the preconditioner (1) have been discussed briefly in section 3. Here they are defined as follows:

- B_I is defined by multigrid or full multigrid, respectively. We employ the V cycle with one ω Jacobi pre- and post-smoothing sweep.
- C_I is defined by the W multigrid cycle with two Gauss Seidel pre- and post-smoothing sweeps.
- For C_C the Dryja preconditioner [5] and the BPS preconditioner (cf. [2]) are utilized in model problem 1 and 2, respectively.

Note that (in contrast to the model problem analysis) the multigrid algorithms use the usual FEM interpolation and restriction between successive grids.

In our tests we compare the multigrid and the full multigrid method for defining the basis transformation B_I. Stimulated by the results of the full two-grid operator different parameters of the ω-Jacobi smoother are investigated.

Additionally we apply exact solvers $B_I = C_I := K_I$. Then C_C solely influences the preconditioner C, and the iteration numbers obtained here thus measure the spectral equivalence constants of C_C to S_C.

The MG and FMG methods are performed on 2 to 8 nested grids. In our numerical tests two different types of triangulation are considered for either model problem. All four coarsest grids

are depicted in Figures 4 and 5. The standard triangulation basically confirms the theoretical analysis but the criss-cross pattern illustrates the strong influence of the triangulation on the iteration numbers and the optimal smoothing parameter ω.

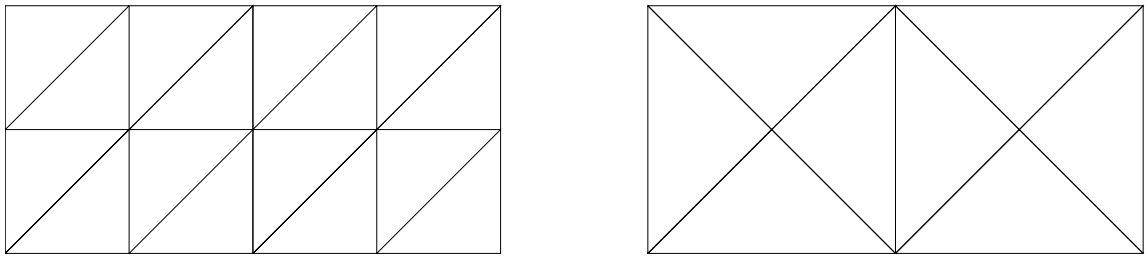

Figure 4: Coarse grids of model problem 1, standard and criss-cross triangulation

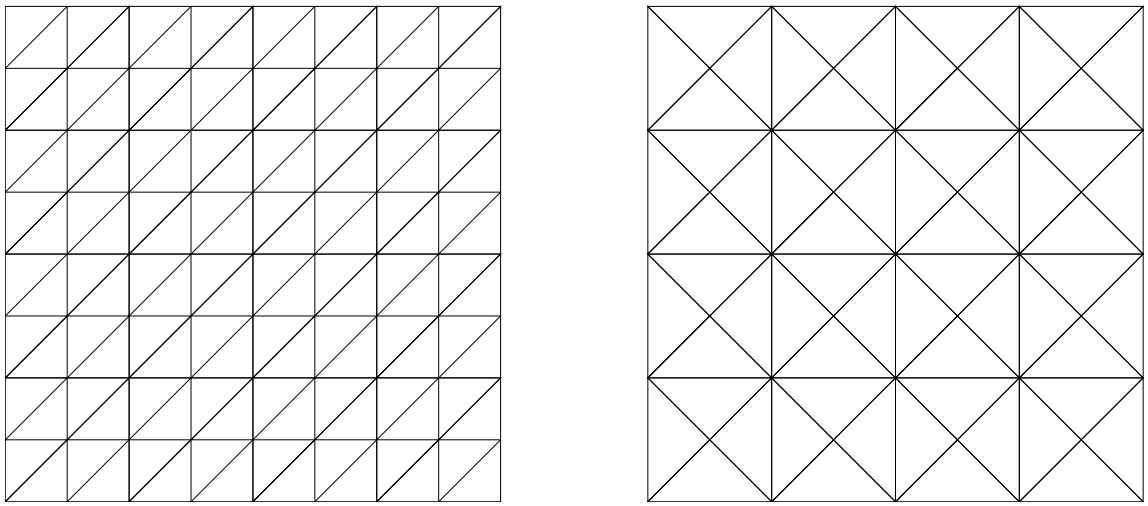

Figure 5: Coarse grids of model problem 2, standard and criss-cross triangulation

5.2.1 Model problem 1, standard triangulation

In the second column of Table 3 we consider the special case of exact solvers ($B_I = C_I = K_I$). The bounded iteration numbers confirm the spectral equivalence of C_C to S_C.

In all other columns the three components B_I, C_I, and C_C are chosen as described above. The iteration numbers are compared for the MG and FMG basis transformation, respectively, and for different smoothing parameters ω of B_I. We observe that the cg iteration numbers depend heavily on the proper choice of ω. The best iteration numbers are obtained with $\omega \approx 1.2 \ldots 1.5$. This corresponds to the theoretical results of the full *two-grid* operator (cf. Figure 3). Thus the theoretical results obtained there can be generalized (to some extent) to the full *multigrid* operator.

For a fixed $\omega \leq 1.5$ the comparison reveals that FMG requires less (or no more) iterations than multigrid. Finally, Table 4 summarized the degrees of freedom and the computational time for $\omega = 1.5$.

	B_I defined by multigrid						
# Grids	$B_I = C_I$	smoothing parameter ω					
	$= K_I$	0.5	0.8	1.0	1.3	1.5	1.7
2	3	6	5	5	6	7	8
3	5	8	7	7	7	9	10
4	5	10	9	8	8	9	11
5	5	15	11	9	8	10	12
6	4	21	14	11	9	10	12
7	4	32	20	14	10	10	12
8	3	50	30	20	12	10	12

	B_I defined by full multigrid						
# Grids	$B_I = C_I$	smoothing parameter ω					
	$= K_I$	0.5	0.8	1.0	1.3	1.5	1.7
2	3	6	5	5	6	7	7
3	5	8	7	7	7	8	10
4	5	10	8	7	7	9	12
5	5	13	9	8	8	9	13
6	4	18	11	9	8	9	14
7	4	24	14	10	8	9	15
8	3	35	19	13	8	9	16

Table 3: Model problem 1, standard triangulation, # cg iterations

5.2.2 Model problem 1, criss-cross triangulation

Here we present a similar comparison as above and draw similar conclusions.

For a fixed $\omega \leq 1.2$ full multigrid performs better (or not worse) than multigrid in terms of iteration numbers. Table 6 summarized the degrees of freedom and the computational time for $\omega = 1.0$.

The theoretical results of the full two-grid operator cannot be extended to this case since the whole analysis is based on the *standard* triangulation (which, in turn, implies a different stiffness matrix). The numerical results strengthen this point since the optimal smoothing parameter $\omega = 1.0$ is smaller than with the standard triangulation.

A comparison of the iteration numbers for both types of the triangulation reveals its strong dependence on the coarse grid and on the smoothing parameter ω.

5.2.3 Model problem 2, standard triangulation

The increasing iteration numbers of the second column of Table 7 (exact solvers $B_I = C_I = K_I$) confirm the growing spectral equivalence constants of the BPS preconditioner C_C to S_C.

For the comparison of MG and FMG, and for different ω, there hold exactly the same conclusions as for the corresponding model problem 1. Table 8 comprises the degrees of freedom and the computational time for $\omega = 1.5$.

# Grids	DoF	B_I by multigrid # iterations	time	B_I by full multigrid # iterations	time
2	45	7	0.1	7	0.1
3	153	9	0.1	8	0.1
4	561	9	0.1	9	0.1
5	2 145	10	0.5	9	0.5
6	8 385	10	2.3	9	2.5
7	33 153	10	9.7	9	10.6
8	131 841	10	39.8	9	44.3

Table 4: Model problem 1, standard triangulation, # cg iterations and time for $\omega = 1.5$

5.2.4 Model problem 2, criss-cross triangulation

The iteration numbers of the MG and FMG method are presented in Table 9. Again the same conclusions as for the corresponding model problem 1 can be drawn. Table 10 comprises the degrees of freedom and the computational time for $\omega = 1.0$.

The strong dependence of the iteration numbers on the triangulation and the smoothing parameter ω is even more obvious for this model problem 2.

6 Summary

Aim of this work was the analysis of efficient, cheap and parallel preconditioners C based on domain decomposition and especially on the Additive Schwarz Method. Starting with known theoretical and practical results the importance of the occurring basis transformation B_I has been pointed out in section 3. The influence of B_I on $\mu = \varrho(S_C^{-1}T_C)$ and thus on $\kappa(C^{-1}K)$ and the number of cg iterations has been shown. Our investigation has been focused on the full multigrid method in order to define B_I, with special emphasis on the behaviour of $\mu = \varrho(S_C^{-1}T_C)$ as $h \to 0$.

The model problem analysis of section 4 could only be carried out for the full *two-grid* operator and the model problem 1. The numerical analysis suggested strongly $\mu = O(h^{-1})$. A proof has been established for a smoothing parameter $\omega = 1.0$.

The first part of the numerical experiments verified our analysis. In the second part the full multigrid and the multigrid basis transformation were compared for both model problems. As anticipated the cg iteration numbers grew slower with the first method. Additionally the strong dependence of the iteration numbers on the triangulation and the smoothing parameter ω has

	B_I defined by multigrid							
# Grids	$B_I = C_I$	smoothing parameter ω						
	$= K_I$	0.5	0.8	0.9	1.0	1.1	1.2	1.5
2	2	4	3	3	3	3	4	4
3	2	6	5	5	5	5	6	7
4	3	8	6	6	6	6	7	11
5	3	9	7	6	6	6	8	15
6	4	12	7	7	6	7	8	21
7	3	15	7	7	6	7	9	32
8	3	20	8	7	6	7	11	49

	B_I defined by full multigrid							
# Grids	$B_I = C_I$	smoothing parameter ω						
	$= K_I$	0.5	0.8	0.9	1.0	1.1	1.2	1.5
2	2	4	3	3	3	3	4	4
3	2	6	5	5	5	5	6	7
4	3	7	6	5	5	6	7	11
5	3	8	5	5	5	6	8	18
6	4	9	5	5	5	6	8	27
7	3	11	5	5	5	6	9	45
8	3	13	5	5	4	7	10	73

Table 5: Model problem 1, criss-cross triangulation, # cg iterations

been pointed out.

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# Grids	DoF	B_I by multigrid # iterations	time	B_I by full multigrid # iterations	time
2	17	3	0.1	3	0.1
3	59	5	0.1	5	0.1
4	239	6	0.1	5	0.1
5	983	6	0.2	5	0.2
6	4 007	6	0.7	5	0.7
7	16 199	6	3.0	5	3.2
8	65 159	6	12.7	4	11.2

Table 6: Model problem 1, criss-cross triangulation, # cg iterations and time for $\omega = 1.0$

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	B_I defined by multigrid						
# Grids	$B_I = C_I$	smoothing parameter ω					
	$= K_I$	0.5	0.8	1.0	1.3	1.5	1.7
2	10	16	12	12	12	15	18
3	11	25	19	16	15	17	22
4	11	39	28	22	17	19	23
5	12	67	41	30	21	21	25
6	13	116	61	45	27	25	27
7	14	199	101	64	35	28	30
8	15	—	170	102	49	35	32

	B_I defined by full multigrid						
# Grids	$B_I = C_I$	smoothing parameter ω					
	$= K_I$	0.5	0.8	1.0	1.3	1.5	1.7
2	10	15	12	11	12	14	18
3	11	24	16	13	13	16	22
4	11	33	22	16	15	18	27
5	12	48	29	22	17	19	30
6	13	75	40	28	19	20	33
7	14	124	60	37	22	21	37
8	15	194	96	56	27	23	44

Table 7: Model problem 2, standard triangulation, # cg iterations

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# Grids	DoF	B_I by multigrid # iterations	time	B_I by full multigrid # iterations	time
2	289	15	0.2	14	0.2
3	1 089	17	0.2	16	0.2
4	4 225	19	0.4	18	0.4
5	16 641	21	1.2	19	1.2
6	66 049	25	5.5	20	5.4
7	263 169	28	25.9	21	24.0
8	1 050 625	35	130.6	23	107.1

Table 8: Model problem 2, standard triangulation, # cg iterations and time for $\omega = 1.5$

# Grids	B_I defined by multigrid							
	$B_I = C_I$	smoothing parameter ω						
	$= K_I$	0.5	0.8	0.9	1.0	1.1	1.2	1.5
2	8	10	8	9	9	10	10	13
3	9	13	11	11	11	12	15	30
4	10	19	12	12	12	14	18	47
5	10	25	14	13	14	16	23	75
6	12	38	18	15	14	18	27	119
7	13	58	21	17	16	20	35	196
8	14	95	26	19	17	24	47	—

# Grids	B_I defined by full multigrid							
	$B_I = C_I$	smoothing parameter ω						
	$= K_I$	0.5	0.8	0.9	1.0	1.1	1.2	1.5
2	8	10	8	9	9	10	10	13
3	9	13	11	11	11	12	15	30
4	10	16	11	11	12	14	18	71
5	10	19	13	12	12	16	22	122
6	12	27	14	14	14	17	27	—
7	13	37	15	15	15	19	34	—
8	14	55	17	17	16	23	45	—

Table 9: Model problem 2, criss-cross triangulation, # cg iterations

# Grids	DoF	B_I by multigrid		B_I by full multigrid	
		# iterations	time	# iterations	time
2	129	9	0.1	9	0.1
3	497	11	0.1	11	0.1
4	2 001	12	0.2	12	0.2
5	8 081	14	0.5	12	0.5
6	32 529	14	1.6	14	1.9
7	130 577	16	7.5	15	8.6
8	523 281	17	32.7	16	38.1

Table 10: Model problem 2, criss-cross triangulation, # cg iterations and time for $\omega = 1.0$