# A Mixed Variational Formulation for 3D Linear and Nonlinear Magnetostatics in the Space $H_0(\mathrm{rot}) \cap H(\mathrm{div})$

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**Abstract**

In the present paper, mixed variational formulations that consider the problems of gauging and interface conditions in a quite natural way are analyzed for magnetostatics. Standard results with respect to the Ladyženskaya-Babuška-Brezzi (LBB) condition can be carried over from Stokes problems to electro-magnetic fields. Nonlinear problems are analyzed and the existence of a unique solution is shown under appropriate assumptions.

*Keywords:* Nonlinear Magnetostatic Problems, Mixed Problems

## 1 Introduction

In the recent years, growing computer resources have allowed to perform three-dimensional electro-magnetic field calculations [15, 16, 18]. One of the essential problems in 3D electro-magnetics is the necessity of gauging. An additional gauging condition has to be imposed on the magnetic vector potential to ensure its uniqueness. There are different gauging strategies considered (see, e.g., [12, 13, 14]). Sometimes, even an ungauged formulation (that may result in an indefinite system of linear equations) is solved by iterative methods [11]. Another strategy uses a well-defined iteration process to approximate the solution of an ungauged formulation [17].

¿From the mathematical point of view, we have to choose appropriate spaces. In [9], the first author proposed a mixed variational formulation in $(H_0^1(\Omega)^3, L^2(\Omega))$. Indeed, existence and uniqueness results as well as discretization error estimates for classes of simple nodal elements (e.g. the Mini element) have been proved. Unfortunately, the mixed variational formulation does not correspond completely with the magnetostatic problem. Moreover, it has been stated in [5] that magnetic field problems should rather be formulated in $H(\mathrm{rot}) \cap H(\mathrm{div})$ instead of $H^1$.

Therefore, the aim of the present paper is to establish a mixed variational formulation for linear and nonlinear magnetostatic problems in the space $H(\mathrm{rot}) \cap H(\mathrm{div})$. Here, we formulate the flux-surface boundary condition as a Dirichlet condition with respect to the tangential components of the vector potential only, and a Neumann condition with respect to its normal component. The Dirichlet condition has been proposed, e.g., in [2], whereas the Neumann condition was not explicitly stated therein, but implicitly contained in the formulation. Thus, we

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set up the problem in the space $H_0(\mathrm{rot}) \cap H(\mathrm{div})$. Other boundary conditions can be formulated in a similar manner.

As in the 2D case, cf. [8], additional assumptions with respect to the coefficient allow a rigorous analysis of nonlinear problems. In particular, the coefficient depends on the solution itself. Assumptions which are physically meaningful and that guarantee unique solveability are stated.

The rest of the paper is organized as follows. In Section 2, we discuss the physical model. We present the variational formulation for linear magnetostatic problems in Section 3 and give analytical results. Section 4 is devoted to the results for nonlinear magnetostatic problems. Finally, we add some concluding remarks in Section 5.

# 2 Physical Model for Magnetostatics

## 2.1 Maxwell equations

Let us first recall the principal equations. We get from the Maxwell equations in the magnetostatic case the relations

$$\begin{aligned} \mathbf{rot}\,\vec{\mathrm{H}} &= \vec{\mathrm{J}} &(1)\\ \vec{\mathrm{H}} &= \nu(|\vec{\mathrm{B}}|)\vec{\mathrm{B}} &(2)\\ \mathrm{div}\,\vec{\mathrm{B}} &= 0 &(3) \end{aligned}$$

with $\nu(|\vec{\mathrm{B}}|) = 1/\mu_0\mu_r(|\vec{\mathrm{B}}|)$ in the nonlinear model, cf. [8]. Here, the magnetic field strength is denoted by $\vec{\mathrm{H}}$, the magnetic induction by $\vec{\mathrm{B}}$, and $\vec{\mathrm{J}}$ is the current density. Since (3) holds, we can introduce the magnetic vector potential $\vec{\mathrm{A}}$ by

$$\mathbf{rot}\,\vec{\mathrm{A}} = \vec{\mathrm{B}}. \tag{4}$$

In magnetostatics, the Coulomb gauge

$$\mathrm{div}\,\vec{\mathrm{A}} = 0 \tag{5}$$

is standard. Thus, we end up with the equations

$$\begin{aligned} \mathbf{rot}\,\left(\nu\left(|\,\mathbf{rot}\,\vec{\mathrm{A}}|\right)\,\mathbf{rot}\,\vec{\mathrm{A}}\right) &= \vec{\mathrm{J}} &(6)\\ \mathrm{div}\,\vec{\mathrm{A}} &= 0 &(7) \end{aligned}$$

for the unknown vector potential $\vec{\mathrm{A}}$. If permanent magnets are involved, $\vec{\mathrm{J}}$ must be replaced by $\vec{\mathrm{J}} + \mathbf{rot}\,\vec{\mathrm{H}}_0$.

According to material properties, the coefficient $\nu$ depends on the position $x$. With the unknown $\mathbf{u} = \vec{\mathrm{A}}$, equations (6) and (7) can be rewritten as

$$\begin{aligned} \mathbf{rot}\,\left(\nu\,(x, |\,\mathbf{rot}\,\mathbf{u}|)\,\mathbf{rot}\,\mathbf{u}\right) &= \mathbf{f} &(8)\\ \mathrm{div}\,\mathbf{u} &= 0. &(9) \end{aligned}$$

We can formulate a *linear problem* by the equations (10) and (9),

$$\mathbf{rot}\,\left(\nu\,(x)\,\mathbf{rot}\,\mathbf{u}\right) = \mathbf{f}, \tag{10}$$

which is a good approximation for the nonlinear problem if we can neglect saturation effects in ferromagnetic materials.

## 2.2 Boundary conditions

The physical model would lead us to a problem in an infinite domain with Sommerfeld's radiation condition, cf. [10]. However, in many practical cases we may assume that the field lines of the magnetic induction $\vec{\mathrm{B}}$ do not enter or leave a bounded domain $\Omega \subset \mathbf{R}^3$ representing the device (e.g. the machine). The latter condition, called flux-surface condition, requires that the normal component vanishes on the boundary $\Gamma = \partial\Omega$ which is assumed to be sufficiently smooth, i.e.

$$\vec{\mathrm{B}}_{\mathrm{n}} = 0 \quad \text{on} \quad \Gamma = \partial\Omega. \tag{11}$$

Suppose for the moment that the normal direction is in the z direction of the Cartesian coordinate system. Then we have

$$\begin{aligned} \vec{\mathrm{B}}_{\mathrm{z}} &= (\mathbf{rot}\,\vec{\mathrm{A}})_{\mathrm{z}} \\ &= \frac{\partial}{\partial \mathrm{x}}\vec{\mathrm{A}}_{\mathrm{y}} - \frac{\partial}{\partial \mathrm{y}}\vec{\mathrm{A}}_{\mathrm{x}}, \end{aligned}$$

and $\vec{\mathrm{B}}_{\mathrm{z}} = 0$ is an immediate consequence of

$$\vec{\mathrm{A}}_{\mathrm{x}} = \vec{\mathrm{A}}_{\mathrm{y}} = \text{ const } = 0.$$

Therefore, we will demand that the tangential components of $\vec{\mathrm{A}}$ vanish on $\Gamma$, i.e.

$$\begin{aligned} \vec{\mathrm{A}}_{\mathrm{t}} = \mathbf{0} \quad &\text{on} \quad \Gamma = \partial\Omega, \\ \vec{\mathrm{A}} \times \mathbf{n} = \mathbf{0} \quad &\text{on} \quad \Gamma = \partial\Omega. \end{aligned} \tag{12}$$

If (12) holds on a smooth surface, then we get

$$\begin{aligned} \operatorname{div}\vec{\mathrm{A}} &= \frac{\partial\vec{\mathrm{A}}_{\mathrm{t}_1}}{\partial \mathrm{t}_1} + \frac{\partial\vec{\mathrm{A}}_{\mathrm{t}_2}}{\partial \mathrm{t}_2} + \frac{\partial\vec{\mathrm{A}}_{\mathrm{n}}}{\partial \mathrm{n}}, \\ \operatorname{div}\vec{\mathrm{A}} &= \frac{\partial\vec{\mathrm{A}}_{\mathrm{n}}}{\partial \mathrm{n}}, \end{aligned} \tag{13}$$

if $\mathrm{t}_1$ and $\mathrm{t}_2$ denote two perpendicular tangential unit vectors. Thus, in view of (5), we should formulate a Neumann condition

$$\frac{\partial\vec{\mathrm{A}}_{\mathrm{n}}}{\partial \mathrm{n}} = 0 \quad \text{on} \quad \Gamma = \partial\Omega \tag{14}$$

for the normal component rather than a Dirichlet condition.

Before we discuss the interface conditions, we perform the standard procedure for getting a variational formulation from (8), i.e. we multiply with a test function $\mathbf{v}$ and obtain

$$\begin{aligned} \breve{a}(\mathbf{u},\mathbf{v}) &= \int_\Omega \mathbf{v}\cdot\,\mathbf{rot}\,(\nu\,\mathbf{rot}\,\mathbf{u})dx \\ &= \int_\Omega (\nu\,\mathbf{rot}\,\mathbf{u})\cdot(\mathbf{rot}\,\mathbf{v})\;dx + \int_\Omega \operatorname{div}\,((\nu\,\mathbf{rot}\,\mathbf{u})\times\mathbf{v})\;dx \\ &= \int_\Omega (\nu\,\mathbf{rot}\,\mathbf{u})\cdot(\mathbf{rot}\,\mathbf{v})\;dx + \int_\Gamma (\nu\,\mathbf{rot}\,\mathbf{u})\times\mathbf{v}\;d\Gamma. \end{aligned} \tag{15}$$

The second integral will be zero if $\mathbf{v}\times\mathbf{n}|_\Gamma = 0$, i.e. $\mathbf{v}_{\mathrm{t}}|_\Gamma = 0$. The latter will be satisfied if $\mathbf{v}$ is taken from an appropriate test space related to $\mathbf{u}$, see Subsection 3.1.

### 2.3 Interface conditions

Let us assume that there are different material coefficients (e.g., for iron and air) in $\Omega$, i.e., without loss of generality we have an interface $\Gamma_{\mathrm{I}}$ and

$$\begin{aligned} \bar{\Omega} &= \bar{\Omega}_1 \cup \bar{\Omega}_2, \quad \text{with} \quad \Omega_1 \cap \Omega_2 = \emptyset, \\ \Gamma_{\mathrm{I}} &= \bar{\Omega}_1 \cap \bar{\Omega}_1, \quad \text{with} \quad \mathrm{meas}(\Gamma_{\mathrm{I}}) > 0, \\ \nu(x) &= \nu^{(i)} \quad \text{for} \quad x \in \Omega_i, \quad i = 1, 2, \end{aligned}$$

and we find (classical) solutions of (6), $\vec{\mathrm{A}}^1$ in $\Omega_1$ and $\vec{\mathrm{A}}^2$ in $\Omega_2$.

The physical interface conditions are well known, it is observed that the normal component of $\vec{\mathrm{B}}$ and the tangential component of $\vec{\mathrm{H}}$ are continuous. We will demonstrate that these conditions are quite naturally included in a variational formulation using a continuous vector potential $\vec{\mathrm{A}}$. On one hand,

$$\vec{\mathrm{A}}_{\mathrm{t}}^1 = \vec{\mathrm{A}}_{\mathrm{t}}^2 \quad \text{on} \quad \Gamma_{\mathrm{I}}$$

implies the "strong" first interface condition

$$\vec{\mathrm{B}}_{\mathrm{n}}^1 = \vec{\mathrm{B}}_{\mathrm{n}}^2 \quad \text{on} \quad \Gamma_{\mathrm{I}}.$$

On the other hand, it follows from

$$\vec{\mathrm{H}}_{\mathrm{t}}^1 = \vec{\mathrm{H}}_{\mathrm{t}}^2 \quad \text{on} \quad \Gamma_{\mathrm{I}}$$

that

$$\begin{aligned} \mathbf{n}_1 \times \vec{\mathrm{H}}^1 &= -\mathbf{n}_2 \times \vec{\mathrm{H}}^2 \\ \left(\mathbf{n}_1 \times \vec{\mathrm{H}}^1\right) \cdot \mathbf{v} &= \left(-\mathbf{n}_2 \times \vec{\mathrm{H}}^2\right) \cdot \mathbf{v} \\ \mathbf{n}_1 \cdot \left(\vec{\mathrm{H}}^1 \times \mathbf{v}\right) &= -\mathbf{n}_2 \cdot \left(\vec{\mathrm{H}}^2 \times \mathbf{v}\right) \\ \int_{\Gamma_{\mathrm{I}}} \vec{\mathrm{H}}^1 \times \mathbf{v} \, d\Gamma_{\mathrm{I}} &+ \int_{\Gamma_{\mathrm{I}}} \vec{\mathrm{H}}^2 \times \mathbf{v} \, d\Gamma_{\mathrm{I}} = 0 \\ \int_{\Gamma_{\mathrm{I}}} \left(\nu^{(1)} \,\mathbf{rot}\, \vec{\mathrm{A}}^1\right) \times \mathbf{v} \, d\Gamma_{\mathrm{I}} &+ \int_{\Gamma_{\mathrm{I}}} \left(\nu^{(2)} \,\mathbf{rot}\, \vec{\mathrm{A}}^2\right) \times \mathbf{v} \, d\Gamma_{\mathrm{I}} = 0, \end{aligned}$$

and the line integrals over the interface will cancel out, i.e., the second interface condition is contained in a "weak" sense in the variational formulation. Thus, the variational formulation presented in the following is the correct representation of the physical behaviour.

We remark that the variational formulation can be derived from the energy functional of the magnetic field,

$$W = \frac{1}{2} \int_{\Omega} \vec{\mathrm{B}} \cdot \vec{\mathrm{H}} \, dx$$

too, and that the second interface condition is not included so naturally if a $-\Delta\vec{\mathrm{A}} = \vec{\mathrm{J}}$ formulation is used.

## 3 Variational formulation for linear problems

### 3.1 Spaces

In the following, we always assume that $\Gamma = \partial\Omega$ and all interfaces between different materials are sufficiently smooth (e.g. piecewise from $C^2$). We use the standard Sobolev spaces (cf. e.g., [7]) $L^2(\Omega)$ and $L^2(\Omega)^3$ equipped with the norm $||.||_0$ and the scalar product $(.,.)_0$, and the space

$$L^2_\star(\Omega) = \left\{ \lambda \in L^2(\Omega) : (\lambda, 1)_0 = \int_{\Omega} \lambda \, dx = 0 \right\}.$$

Next, we define the space $H^1(\Omega)^3$ with

$$|\mathbf{v}|_1^2 = ||\,\mathbf{grad}\,\mathbf{v}||_0^2 = \int_\Omega \mathbf{v}_{i,j}\mathbf{v}_{i,j}\ dx,$$

where $\mathbf{v}_{i,j} = \frac{\partial \mathbf{v}_i}{\partial x_j}$ and Einstein's summation convention is applied to indices $i$ and $j$, and

$$||\mathbf{v}||_1^2 = |\mathbf{v}|_1^2 + ||\mathbf{v}||_0^2.$$

It is a standard result that $|.|_1$ is a norm of $H_0^1(\Omega)^3$.

The following lemma (cf. [6]) allows us to represent the norm $|.|_1$ of the $H_0^1(\Omega)^3$.

**Lemma 1** *For all $\mathbf{v} \in H_0^1(\Omega)^3$ the equation*

$$|\mathbf{v}|_1^2 = ||\,\mathbf{rot}\,\mathbf{v}||_0^2 + ||\,\mathrm{div}\,\mathbf{v}||_0^2 \tag{16}$$

*holds.*

**Proof.** Assume $\mathbf{v} \in H^2(\Omega)^3$. We rewrite the **rot** operator as follows

$$\begin{aligned}
\int_\Omega \mathbf{rot}\,\mathbf{v}\cdot\mathbf{rot}\,\mathbf{v}\ dx &= \int_\Omega \mathbf{v}_{1,2}\,(\mathbf{v}_{1,2}-\mathbf{v}_{2,1}) + \mathbf{v}_{2,1}\,(\mathbf{v}_{2,1}-\mathbf{v}_{1,2}) + \ldots\ dx \\
&= \int_\Omega \mathbf{v}_{i,j}\,(\mathbf{v}_{i,j}-\mathbf{v}_{j,i})\ dx \\
&= -\int_\Omega \mathbf{v}_i\,(\mathbf{v}_{i,j}-\mathbf{v}_{j,i})_j\ dx \\
&= -\int_\Omega \mathbf{v}_i\,(\mathbf{v}_{i,jj}-\mathbf{v}_{j,ij})\ dx \\
&= -\int_\Omega \mathbf{v}_i\,(\mathbf{v}_{i,jj}-\mathbf{v}_{j,ji})\ dx \\
&= -\int_\Omega \mathbf{v}_i\,(\mathbf{v}_{i,jj}-(\,\mathrm{div}\,\mathbf{v})_i)\ dx \\
&= \int_\Omega \mathbf{v}_{i,j}\mathbf{v}_{i,j} - (\,\mathrm{div}\,\mathbf{v})^2\ dx,
\end{aligned}$$

and after having applied partial integration twice we arrive at

$$||\,\mathbf{rot}\,\mathbf{v}||_0^2 = |\mathbf{v}|_1^2 - ||\,\mathrm{div}\,\mathbf{v}||_0^2 \quad \forall \mathbf{v} \in H_0^1(\Omega)^3 \cap H^2(\Omega)^3. \tag{17}$$

Since $H_0^1(\Omega)^3 \cap H^2(\Omega)^3$ is dense in $H_0^1(\Omega)^3$, equation (16) holds for all $\mathbf{v} \in H_0^1(\Omega)^3$. ■

Let us discuss further spaces, cf. [4]. We define

$$\begin{aligned}
H(\,\mathrm{div}\,;\Omega) &= \left\{\mathbf{v} \in L^2(\Omega)^3 : \,\mathrm{div}\,\mathbf{v} \in L^2(\Omega)\right\}, \\
H(\,\mathbf{rot}\,;\Omega) &= \left\{\mathbf{v} \in L^2(\Omega)^3 : \,\mathbf{rot}\,\mathbf{v} \in L^2(\Omega)^3\right\}, \\
||\mathbf{v}||_{\mathrm{div}}^2 &= ||\mathbf{v}||_0^2 + ||\,\mathrm{div}\,\mathbf{v}||_0^2, \\
||\mathbf{v}||_{\mathbf{rot}}^2 &= ||\mathbf{v}||_0^2 + ||\,\mathbf{rot}\,\mathbf{v}||_0^2, \\
H_0(\,\mathrm{div}\,;\Omega) &= \overline{\mathcal{D}(\Omega)}^{H(\,\mathrm{div}\,;\Omega)}, \\
H_0(\,\mathbf{rot}\,;\Omega) &= \overline{\mathcal{D}(\Omega)}^{H(\,\mathbf{rot}\,;\Omega)}.
\end{aligned}$$

It is proved in [7] that

$$H_0^1(\Omega)^3 = H_0(\,\mathrm{div}\,;\Omega) \cap H_0(\,\mathbf{rot}\,;\Omega).$$

We remark that $H_0(\operatorname{div};\Omega)$ has a zero trace of the normal component on $\Gamma = \partial\Omega$, whereas $H_0(\mathbf{rot};\Omega)$ has a zero trace of the tangential component.

Therefore, the space $H_0(\mathbf{rot};\Omega)$ seems to be appropriate to the flux line boundary condition, cf. (12). Since we are interested in formulating a gauge condition that involves the divergence of the vector potential, we introduce the space

$$\mathbf{Y} = H_0(\mathbf{rot};\Omega) \cap H(\operatorname{div};\Omega) \tag{18}$$

equipped with the norm

$$||\mathbf{v}||_{\mathbf{Y}}^2 = ||\mathbf{v}||_0^2 + ||\,\mathbf{rot}\,\mathbf{v}||_0^2 + ||\operatorname{div}\mathbf{v}||_0^2. \tag{19}$$

¿From [7], we have the following results.

**Lemma 2** *Assume that $\Omega$ is bounded, Lipschitz-continuous, simply-connected and $\Gamma$ has just one component. Then there exists a positive constant $c$ such that*

$$||\mathbf{v}||_0 \le c\,(||\,\mathbf{rot}\,\mathbf{v}||_0 + ||\operatorname{div}\mathbf{v}||_0) \quad \forall \mathbf{v} \in \mathbf{Y}$$

*holds.*

**Proof.** See [7], Lemma 3.4. ■

**Lemma 3** *Assume that $\Omega$ is bounded, Lipschitz-continuous, simply-connected and $\Gamma$ has just one component. Then, the seminorm*

$$|\mathbf{v}|_{\mathbf{Y}} = \left(||\,\mathbf{rot}\,\mathbf{v}||_0^2 + ||\operatorname{div}\mathbf{v}||_0^2\right)^{1/2} \tag{20}$$

*is a norm in $\mathbf{Y}$, too.*

**Proof.** Lemma 3 is a direct consequence of Lemma 2. ■

The next lemma requires more than the standard assumptions for technical magnetic field problems. Indeed, electromagnetic devices may have re-entrant corners. Therefore, we will not base our further investigations on Lemma 4.

**Lemma 4** *Assume further that either $\Omega$ has a $\mathcal{C}^{1,1}$ boundary, or $\Omega$ is a convex polyhedron. Then $\mathbf{Y}$ is continuously imbedded in $H^1(\Omega)^3$.*

**Proof.** See [7], Theorem 3.7. ■

### 3.2 The mixed variational formulation

We establish a mixed variational formulation of the linear problem in the pair $(\mathbf{Y}, L^2_\star(\Omega))$ of spaces. Let be

$$\begin{aligned} \mathbf{Y} &= H_0(\mathbf{rot};\Omega) \cap H(\operatorname{div};\Omega), \\ \mathbf{M} &= L^2_\star(\Omega), \quad ||.||_{\mathbf{M}} = ||.||_0 \end{aligned}$$

where the space $\mathbf{Y}$ is equipped with the norm $|.|_{\mathbf{Y}}$, cf. (20). Using a trick similar to that in [1] we choose a constant $\rho =$ const $> 0$ and define

$$\hat{a}(\mathbf{u},\mathbf{v}) = \int_\Omega \nu\,\mathbf{rot}\,\mathbf{u} \cdot \mathbf{rot}\,\mathbf{v} + \rho \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, dx \tag{21}$$

$$b(\mathbf{v},\lambda) = -\int_\Omega \lambda \operatorname{div}\mathbf{v}\, dx. \tag{22}$$

Then we may write the standard variational formulation:
**(VF1)** *Find a pair* $(\mathbf{u},\lambda) \in \mathbf{Y} \times L^2_\star(\Omega)$ *such that*

$$\begin{aligned} \hat{a}(\mathbf{u},\mathbf{v}) + b(\mathbf{v},\lambda) &= \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{Y}, &(23)\\ b(\mathbf{u},\mu) &= 0 \quad \forall \mu \in \mathbf{M}. &(24)\end{aligned}$$

## 3.3 The Brezzi theorem

We recall the Brezzi theorem, e.g. from [3].

**THEOREM 1** *Suppose a mixed variational problem in abstract setting*
*Find* $(\mathbf{u},\lambda) \in \mathbf{X} \times \mathbf{M}$ *such that*

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},\lambda) &= \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}, &(25)\\ b(\mathbf{u},\mu) &= \langle g,\mu\rangle \quad \forall \mu \in \mathbf{M}. &(26)\end{aligned}$$

*Assume that the following conditions are satisfied*
**(A1)** $\mathbf{f} \in \mathbf{X}^\star$, $g \in \mathbf{M}^\star$,
**(A2)** $\exists \alpha_2 = \text{const} > 0$ *and* $\beta_2 = \text{const} > 0$ *with*

$$\begin{aligned} |a(\mathbf{u},\mathbf{v})| &\leq \alpha_2 ||\mathbf{u}||_\mathbf{X} ||\mathbf{v}||_\mathbf{X} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{X}\\ |b(\mathbf{v},\mu)| &\leq \beta_2 ||\mathbf{v}||_\mathbf{X} ||\mu||_\mathbf{M} \quad \forall \mathbf{v} \in \mathbf{X}, \forall \mu \in \mathbf{M},\end{aligned}$$

**(A3)** *the LBB (Ladyženskaya-Babuška-Brezzi) condition, i.e.*
$\exists \beta_1 = \text{const} > 0$ *such that*

$$\sup_{\mathbf{v}\in\mathbf{X},\mathbf{v}\neq\mathbf{0}} \frac{b(\mathbf{v},\mu)}{||\mathbf{v}||_\mathbf{X}} \geq \beta_1 ||\mu||_\mathbf{M} \quad \forall \mu \in \mathbf{M}, \tag{27}$$

**(A4)** *the* $\mathbf{V}_0$ *– ellipticity, i.e.* $\exists \alpha_1 = \text{const} > 0$ *with*

$$a(\mathbf{v},\mathbf{v}) \geq \alpha_1 ||\mathbf{v}||^2_\mathbf{X} \quad \forall \mathbf{v} \in \mathbf{V}_0 \tag{28}$$

*where*

$$\mathbf{V}_0 = \mathbf{V}(0) = \{\mathbf{v} \in \mathbf{X} : b(\mathbf{v},\mu) = 0 \quad \forall \mu \in \mathbf{M}\}. \tag{29}$$

*Then there exists a unique solution* $(\mathbf{u},\lambda) \in \mathbf{X} \times \mathbf{M}$ *and the a-priori estimates*

$$\begin{aligned} ||\mathbf{u}||_\mathbf{X} &\leq \frac{1}{\alpha_1}||\mathbf{f}||_{\mathbf{X}^\star} + \frac{1}{\beta_1}\left(1 + \frac{\alpha_2}{\alpha_1}\right)||g||_{\mathbf{M}^\star}\\ ||\lambda||_\mathbf{M} &\leq \frac{1}{\beta_1}\left(1 + \frac{\alpha_2}{\alpha_1}\right)||f||_{\mathbf{X}^\star} + \frac{\alpha_2}{\beta_1^2}\left(1 + \frac{\alpha_2}{\alpha_1}\right)||g||_{\mathbf{M}^\star}\end{aligned}$$

*hold.*

**Proof.** See, e.g., [4]. ■

We apply Theorem 1 to the variational formulation **(VF1)**.

**THEOREM 2** *Suppose that* $\mathbf{f} \in \mathbf{Y}^\star$, $\rho = \text{const} > 0$ *and that* $\nu(x)$ *is a piecewise constant function fulfilling*

$$0 < \underline{\nu} \leq \nu(x) \leq \bar{\nu}. \tag{30}$$

*Then (23), (24) has a unique solution* $(\mathbf{u},\lambda) \in \mathbf{Y} \times L^2_\star(\Omega)$ *for which a-priori estimates*

$$\begin{aligned} |\mathbf{u}|_1 &\leq \frac{1}{c_{\mathrm{L}}}\,||\mathbf{f}||_{\mathbf{Y}^\star} \\ ||\lambda||_0 &\leq \frac{1}{\underline{\nu}}(1+\frac{\max(\bar{\nu},\rho)}{c_{\mathrm{L}}})||\mathbf{f}||_{\mathbf{Y}^\star} \end{aligned}$$

*hold.*

**Proof.** We have to verify the assumptions of Theorem 1.
**(A1)** It is the assumption, and $g = 0$.
**(A2)** Applying Lemma 3 we get

$$\begin{aligned} |\hat{a}(\mathbf{u},\mathbf{v})| &= \left|\int_\Omega \nu\,\mathbf{rot}\,\mathbf{u}\cdot\mathbf{rot}\,\mathbf{v}\;dx\right| + \left|\int_\Omega \rho\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\;dx\right| \\ &\leq \bar{\nu}||\,\mathbf{rot}\,\mathbf{u}||_0\;||\,\mathbf{rot}\,\mathbf{v}||_0 + \rho||\,\mathrm{div}\,\mathbf{u}||_0\;||\,\mathrm{div}\,\mathbf{v}||_0 \\ &\leq \max(\bar{\nu},\rho)\;|\mathbf{u}|_{\mathbf{Y}}\;|\mathbf{v}|_{\mathbf{Y}}, \end{aligned}$$

i.e., we can set $\alpha_2 = \max(\bar{\nu},\rho)$, and

$$\begin{aligned} |b(\mathbf{v},\mu)| &= \left|\int_\Omega \mu\,\mathrm{div}\,\mathbf{v}\;dx\right| \\ &\leq ||\mu||_0\;||\,\mathrm{div}\,\mathbf{v}||_0 \\ &\leq ||\mu||_0\;|\mathbf{v}|_{\mathbf{Y}}, \end{aligned}$$

i.e. $\beta_2 = 1$.
**(A3)** The LBB condition has been proved for the Stokes problem, cf. e.g. [6, Theorem 3.7]. There exists a constant $c_{\mathrm{L}} > 0$ with

$$\sup_{\mathbf{v}\in H^1_0(\Omega)^3,\mathbf{v}\neq\mathbf{0}} \frac{\int_\Omega \mu\,\mathrm{div}\,\mathbf{v}\;dx}{|\mathbf{v}|_1} \geq c_{\mathrm{L}}||\mu||_0 \quad \forall\mu\in L^2_\star(\Omega). \tag{31}$$

¿From Lemma 1 and (20) we get $|\mathbf{v}|_{\mathbf{Y}} = |\mathbf{v}|_1\ \forall\mathbf{v}\in H^1_0(\Omega)^3$, and with $H^1_0(\Omega)^3\subset\mathbf{Y}$ it follows that

$$\sup_{\mathbf{v}\in\mathbf{Y},\mathbf{v}\neq\mathbf{0}} \frac{\int_\Omega \mu\,\mathrm{div}\,\mathbf{v}\;dx}{|\mathbf{v}|_{\mathbf{Y}}} \geq \sup_{\mathbf{v}\in H^1_0(\Omega)^3,\mathbf{v}\neq\mathbf{0}} \frac{\int_\Omega \mu\,\mathrm{div}\,\mathbf{v}\;dx}{|\mathbf{v}|_{\mathbf{Y}}} \geq c_{\mathrm{L}}||\mu||_0 \quad \forall\mu\in L^2_\star(\Omega). \tag{32}$$

**(A4)** We can prove ellipticity for the whole space $\mathbf{Y}$, i.e.

$$\begin{aligned} \hat{a}(\mathbf{v},\mathbf{v}) &\geq \underline{\nu}||\,\mathbf{rot}\,\mathbf{v}||_0^2 + \rho||\,\mathrm{div}\,\mathbf{v}||_0^2 \quad \forall\mathbf{v}\in\mathbf{Y} \\ \hat{a}(\mathbf{v},\mathbf{v}) &\geq \min(\underline{\nu},\rho)\;|\mathbf{v}|^2_{\mathbf{Y}} \quad \forall\mathbf{v}\in\mathbf{Y}, \end{aligned} \tag{33}$$

and $\alpha_1 = \min(\underline{\nu},\rho)$. ■

We remark that it may be useful to choose $\rho$ depending on the actual coefficients $\nu$, e.g., $\underline{\nu}\leq\rho\leq\bar{\nu}$.

### 3.4 Equivalence with magnetostatics

Next, we have to prove that the solution $(\mathbf{u},\lambda)$ of the variational problem (23), (24) represents a solution of the magnetostatic problem, too. Indeed, in (23) we have added the terms $\int_\Omega \rho\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\;dx$ and $-\int_\Omega \lambda\,\mathrm{div}\,\mathbf{v}\;dx$ to the formulation of (15). We show that for the solution $(\mathbf{u},\lambda)$ the sum of both terms vanishes. We recall the following theorem from [7].

**THEOREM 3** *Assume that $\Omega$ is bounded, Lipschitz-continuous and simply-connected. A function $\mathbf{v} \in L^2(\Omega)^3$ satisfies*

$$\mathbf{rot}\,\mathbf{v} = \mathbf{0} \quad in\ \Omega$$

*iff there exists a unique function $\dot{p} \in H^1(\Omega)\backslash\mathbf{R}$ such that*

$$\mathbf{u} = \operatorname{grad} \dot{p}.$$

**Proof.** See [7], Theorem 2.9. ∎

In the next theorem, we will assume that the electrical current density $\mathbf{f}$ is divergence-free. Indeed, since electrical charges cannot appear or disappear, this assumption represents the physical behaviour.

**THEOREM 4** *Assume that $\Omega$ is bounded, Lipschitz-continuous and simply-connected. Assume further that $\mathbf{f} \in H(\operatorname{div};\Omega)$ is divergence-free, i.e.,*

$$\operatorname{div} \mathbf{f} = 0. \tag{34}$$

*Then, the unique solution $(\mathbf{u}, \lambda)$ of*

$$\begin{aligned} \int_\Omega \nu\, \mathbf{rot}\,\mathbf{u} \cdot \mathbf{rot}\,\mathbf{v}\, dx + \int_\Omega \rho \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx - \int_\Omega \lambda \operatorname{div} \mathbf{v}\, dx &= \int_\Omega \mathbf{f}\,\mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathbf{Y} && (35)\\ -\int_\Omega \mu \operatorname{div} \mathbf{u}\, dx &= 0 \quad \forall \mu \in \mathbf{M} && (36)\end{aligned}$$

*fulfills*

$$\int_\Omega \nu\, \mathbf{rot}\,\mathbf{u} \cdot \mathbf{rot}\,\mathbf{v}\, dx = \int_\Omega \mathbf{f}\,\mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathbf{Y}. \tag{37}$$

**Proof.** Let us choose an arbitrary $\phi \in L^2(\Omega)$. Then it is well-known that the Dirichlet problem

$$\begin{aligned} \Delta p &= \phi \quad \text{in } \Omega && (38)\\ p &= 0 \quad \text{on } \partial\Omega && (39)\end{aligned}$$

has a unique weak solution $p \in H_0^1(\Omega)$.

¿From Theorem 3 we deduce that there exists a $\mathbf{v} \in L^2(\Omega)^3$ with

$$\mathbf{rot}\,\mathbf{v} = 0 \tag{40}$$

and

$$\mathbf{v} = \operatorname{grad} p. \tag{41}$$

Next, from (39) we get that the tangential derivatives of $p$ vanish, i.e. that the tangential components of $\mathbf{v}$ are zero. Thus, $\mathbf{v} \in H_0(\mathbf{rot};\Omega)$ holds. Further, we can determine the divergence of $\mathbf{v}$ as follows

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \operatorname{div}\,(\operatorname{grad} p)\\ &= \Delta p\\ &= \phi. && (42)\end{aligned}$$

Consequently, $\mathbf{v} \in H(\operatorname{div};\Omega)$ and $\mathbf{v} \in \mathbf{Y}$. We can apply $\mathbf{v}$ as a test function in (35), and we get with (40)

$$\begin{aligned} \int_\Omega \rho \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx - \int_\Omega \lambda \operatorname{div} \mathbf{v}\, dx &= \int_\Omega \mathbf{f}\,\mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathbf{Y}, && (43)\\ \int_\Omega (\rho \operatorname{div} \mathbf{u} - \lambda)\, \phi\, dx &= \int_\Omega \mathbf{f}\, \operatorname{grad} p\, dx, && (44)\\ \int_\Omega (\rho \operatorname{div} \mathbf{u} - \lambda)\, \phi\, dx &= -\int_\Omega \operatorname{div} \mathbf{f} p\, dx + \int_{\partial\Omega} \mathbf{f}\, p\, \mathrm{n}\, ds. && (45)\end{aligned}$$

With the assumptions (39) and (34) on $p$ and $\mathbf{f}$, the right-hand side is zero. Therefore,

$$\int_\Omega (\rho \operatorname{div} \mathbf{u} - \lambda)\, \phi \, dx = 0 \tag{46}$$

holds, and setting $\phi := \operatorname{div} \mathbf{v} \in L^2(\Omega)$ for any $\mathbf{v} \in \mathbf{Y}$, we arrive at (37). ∎

Thus, with (15) we get

$$\int_\Omega (\nu \,\mathbf{rot}\, \mathbf{u}) \cdot (\,\mathbf{rot}\, \mathbf{v})\, dx = \int_\Omega \mathbf{f} \ \mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathbf{Y}, \tag{47}$$

and we conclude the relation

$$\mathbf{rot}\ (\nu \,\mathbf{rot}\, \mathbf{u}) = \mathbf{f} \tag{48}$$

in the weak sense.

We remark that we do not require that $\mathbf{f}\mathrm{n} = \mathbf{0}$ on $\partial\Omega$.

Further, setting $\phi := \lambda \in L^2(\Omega)_\star$, we get from (46) and (36) that

$$\begin{aligned} \int_\Omega (\rho \operatorname{div} \mathbf{u} - \lambda)\, \lambda \, dx &= 0, \\ \int_\Omega \lambda^2 \, dx &= 0, \end{aligned}$$

and $\lambda = 0$. Therefore, we can omit the $\lambda$-term in (35), and solve the coercive problem:
**(VF1$^\star$)** *Find* $\mathbf{u} \in \mathbf{Y}$ *such that*

$$\int_\Omega \nu \,\mathbf{rot}\, \mathbf{u} \cdot \,\mathbf{rot}\, \mathbf{v}\, dx + \int_\Omega \rho \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx = \int_\Omega \mathbf{f}\ \mathbf{v}\, dx \quad \forall \mathbf{v} \in \mathbf{Y}. \tag{49}$$

The unique solution $\mathbf{u}$ of (VF1$^\star$) coincides with the $\mathbf{u}$ in the solution of (VF1). We may discretize and solve the problem in the formulation (VF1$^\star$) instead of (VF1).

# 4 Variational formulation for nonlinear problems

## 4.1 Monotonicity and Lipschitz continuity

In many practical cases, the nonlinear behaviour of ferromagnetic materials cannot be neglected. Therefore, we extend the analysis of 2D monotone, Lipschitz continuous nonlinear problems (see [8]) to three dimensions.

Let $\bar{\Omega}$ be decomposed into subdomains

$$\bar{\Omega} = \bigcup_{j=1}^{N_M} \bar{\Omega}_j, \quad \text{with} \quad \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j$$

representing materials with different magnetic properties. We assume that the function $\nu\,(x, |\,\mathbf{rot}\, \mathbf{u}|)$ in (8) depends on the position $x \in \Omega$, but $\nu$ is always the same function in one of the $\Omega_j$, i.e.,

$$\nu(x, z) = \nu^{(j)}(z) \quad \text{if} \quad x \in \Omega_j. \tag{50}$$

We suppose the following properties of the functions $\nu^{(j)}$, $j = 1, \ldots, N_M$ which are justified by the physical model (cf. [8]):

**(M0)** $\nu^{(j)}(z) = \nu_1^{(j)} = \text{ const.} \quad \forall z \in [0, z_1^{(j)}]$,

**(M1)** $\nu^{(j)}(z) \geq \nu_1^{(j)} \quad \forall z \geq 0$,

**(M2)** $\nu^{(j)}(z)$ is a monotonic increasing function,

**(M3)** $\lim_{z\to\infty} \nu^{(j)}(z) = \nu_\infty^{(j)}$, where $\nu_\infty^{(j)} = (\mu_0)^{-1}$ for ferromagnetic materials,

**(M4)** there exists $\nu^{(j)\prime}(z) \quad \forall\ z\ \geq\ 0$, and there exists a constant $M_1^{(j)}$ with $\nu^{(j)\prime}(z) \leq M_1^{(j)} \quad \forall z \geq 0$,

**(M5)** there exists a constant $M^{(j)}$ with $\nu^{(j)\prime}(z)\ z + \nu^{(j)}(z) \leq M^{(j)} \quad \forall z \geq 0$.

Let $\nu_1, M_1$ and $M$ be the global constants with

$$\begin{aligned} \nu_1 &= \min_{j=1,\dots,N_M} \nu_1^{(j)} && (51)\\ M_1 &= \max_{j=1,\dots,N_M} M_1^{(j)} && (52)\\ M &= \max_{j=1,\dots,N_M} M^{(j)}. && (53) \end{aligned}$$

We define the bivariate form $a : H^1(\Omega)^3 \times H^1(\Omega)^3 \longrightarrow \mathbf{R}$ which is nonlinear in its first argument, but linear in its second, by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nu(x, |\,\mathbf{rot}\,\mathbf{u}|)\,\mathbf{rot}\,\mathbf{u}\cdot\ \mathbf{rot}\,\mathbf{v} + \rho \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\ dx. \qquad (54)$$

Then, the variational formulation for the problem (8), (9) can be written as:
**(VF2)** *Find a pair* $(\mathbf{u},\lambda) \in \mathbf{Y} \times L^2_\star(\Omega)$ *such that*

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},\lambda) &= \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{Y}, && (55)\\ b(\mathbf{u},\mu) &= 0 \quad \forall \mu \in \mathbf{M}, && (56) \end{aligned}$$

where the bilinear form $b$ is the identical as in (22). Now, we prove monotonicity and Lipschitz continuity of the form $a$ resp. the (nonlinear) operator $A : \mathbf{Y} \longrightarrow \mathbf{Y}^\star$ defined by

$$\langle A\mathbf{u},\mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{Y}.$$

**Lemma 5** *Assume* **(M1)** *and* **(M2)**. *Then $A$ is strongly monotone, i.e., the inequality*

$$\langle A\mathbf{u} - A\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \geq \min(\nu_1,\rho)\ |\mathbf{u}-\mathbf{v}|^2_{\mathbf{Y}} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{Y} \qquad (57)$$

*holds.*

**Proof.** The proof is similar to the proof of Lemma 2.1 in [8]. We consider vectors $s,t \in \mathbf{R}^3$, and assume that $|s| \geq |t|$ holds without loss of generality. Then, we get

$$|s|^2 - s\cdot t \geq |s|^2 - |s|\ |t| \geq 0$$

and from **(M1)** and **(M2)** it follows that

$$\begin{aligned} (\nu(|s|)s - \nu(|t|)t)\cdot(s-t) &= \nu(|s|)(|s|^2 - s\cdot t) + \nu(|t|)(|t|^2 - s\cdot t)\\ &\geq \nu(|t|)(|s|^2 - s\cdot t) + \nu(|t|)(|t|^2 - s\cdot t)\\ &= \nu(|t|)|s-t|^2\\ &\geq \nu_1|s-t|^2 \end{aligned}$$

holds. Setting $s = \textbf{rot}\,\mathbf{u}(x)$ and $t = \textbf{rot}\,\mathbf{v}(x)$ and integrating we obtain

$$\int_\Omega (\nu(|\textbf{rot}\,\mathbf{u}|)\,\textbf{rot}\,\mathbf{u} - \nu(|\textbf{rot}\,\mathbf{v}|)\,\textbf{rot}\,\mathbf{v}) \cdot (\textbf{rot}\,\mathbf{u} - \textbf{rot}\,\mathbf{v})\,dx \geq \nu_1 \int_\Omega |\textbf{rot}\,(\mathbf{u}-\mathbf{v})|^2 dx,$$

and with

$$\int_\Omega \rho\,\mathrm{div}\,(\mathbf{u}-\mathbf{v})\,\mathrm{div}\,(\mathbf{u}-\mathbf{v})\;dx = \rho\;||\,\mathrm{div}\,(\mathbf{u}-\mathbf{v})||_0^2 \tag{58}$$

and Lemma 3 we get the estimate (57). ∎

**Lemma 6** *Suppose that* **(M4)** *and* **(M5)** *hold. Then $A$ is Lipschitz-continuous, i.e., the inequality*

$$||A\mathbf{u} - A\mathbf{v}||_{\mathbf{Y}^\star} \leq \max(M,\rho)\;||\mathbf{u}-\mathbf{v}||_{\mathbf{Y}} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{Y} \tag{59}$$

*holds.*

**Proof.** ¿From **(M4)** and **(M5)** we obtain

$$\nu'(z)z + \nu(z) \leq M \quad \forall z \geq 0,$$

and by the mean value theorem we get

$$|\nu(z_1)z_1 - \nu(z_2)z_2| \leq M|z_1 - z_2| \quad \forall z_1, z_2 \geq 0. \tag{60}$$

Consider vectors $s, t \in \mathbf{R}^3$. The same calculation as in the proof of Lemma 2.2 in [8] yields

$$|\nu(|s|)s - \nu(|t|)t| \leq M|s-t| \quad \forall s,t \in \mathbf{R}^3. \tag{61}$$

We set $s = \textbf{rot}\,\mathbf{u}(x)$, $t = \textbf{rot}\,\mathbf{v}(x)$, multiply with $|\,\textbf{rot}\,\mathbf{w}(x)|$, and take the integral over $\Omega$. The result is

$$\int_\Omega |\nu(|\textbf{rot}\,\mathbf{u}|)\;\textbf{rot}\,\mathbf{u} - \nu(|\textbf{rot}\,\mathbf{v}|)\;\textbf{rot}\,\mathbf{v}|\;\;|\textbf{rot}\,\mathbf{w}|\;dx \leq M \int_\Omega |\textbf{rot}\,(\mathbf{u}-\mathbf{v})|\;\;|\textbf{rot}\,\mathbf{w}|\;dx,$$

and with well-known estimates, an equation for the $\rho$ term, and Lemma 3 we obtain

$$\langle A\mathbf{u} - A\mathbf{v}, \mathbf{w}\rangle \leq \max(M,\rho)\;|\mathbf{u}-\mathbf{v}|_{\mathbf{Y}}\;|\mathbf{w}|_{\mathbf{Y}} \quad \forall \mathbf{w} \in \mathbf{Y} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{Y},$$

and the desired estimate (59). ∎

## 4.2 Nonlinear extension of Brezzi's theorem

The following lemma provides equivalent formulations for the LBB condition in an abstract setting.

**Lemma 7** *Define the linear operator $B : \mathbf{X} \longrightarrow \mathbf{M}^\star$ and its adjoint operator $B^\star : \mathbf{M} \longrightarrow \mathbf{X}^\star$ in abstract spaces $\mathbf{X}$, $\mathbf{M}$ by*

$$\langle B\mathbf{u}, \mu\rangle = b(\mathbf{u},\mu) \quad \forall \mu \in \mathbf{M} \quad \forall \mathbf{u} \in \mathbf{X} \tag{62}$$

$$\langle B^\star\lambda, \mathbf{v}\rangle = b(\mathbf{v},\lambda) \quad \forall \mathbf{v} \in \mathbf{X} \quad \forall \lambda \in \mathbf{M}. \tag{63}$$

*Then, the following formulations of the LBB condition* $(1^\circ)$, $(2^\circ)$, $(3^\circ)$ *are equivalent.*
$(1^\circ)$ $\exists \beta_1 = \mathrm{const} > 0$ *such that*

$$\sup_{\mathbf{v}\in\mathbf{X},\mathbf{v}\neq\mathbf{0}} \frac{b(\mathbf{v},\mu)}{||\mathbf{v}||_{\mathbf{X}}} \geq \beta_1 ||\mu||_{\mathbf{M}} \quad \forall \mu \in \mathbf{M}, \tag{64}$$

(2°) *The operator* $B : \mathbf{V_0}^\perp \longrightarrow \mathbf{M}^\star$ *is an isomorphism, and the estimate*

$$||B\mathbf{v}||_{\mathbf{M}^\star} \geq \beta_1 ||\mathbf{v}||_{\mathbf{X}} \quad \forall \mathbf{v} \in \mathbf{V_0}^\perp \tag{65}$$

*holds, where* $\mathbf{V_0}^\perp = \{\mathbf{u} \in \mathbf{X} : (\mathbf{u}, \mathbf{v})_{\mathbf{X}} = 0 \quad \forall \mathbf{v} \in \mathbf{V}_0 \equiv \ker B\}$.
(3°) *The operator* $B^\star : \mathbf{M} \longrightarrow \mathbf{V_0}^0$ *is an isomorphism, and the estimate*

$$||B^\star \mu||_{\mathbf{X}^\star} \geq \beta_1 ||\mu||_{\mathbf{M}} \quad \forall \mu \in \mathbf{M} \tag{66}$$

*holds, where* $\mathbf{V_0}^0 = (\ker B)^0 \subset \mathbf{X}^\star$ *denotes the polar of* $\mathbf{V}_0$, *i.e.*

$$\mathbf{V_0}^0 = \{\mathbf{l} \in \mathbf{X}^\star : \langle \mathbf{l}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathbf{V}_0 \equiv \ker B\}.$$

**Proof.** The lemma is proved, e.g., in [3, Lemma 4.2]. ■

Next, we present an extension of Brezzi's theorem to a class of nonlinear problems.

**THEOREM 5** *Suppose a mixed variational problem in abstract setting*
*Find* $(\mathbf{u}, \lambda) \in \mathbf{X} \times \mathbf{M}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \lambda) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}, \tag{67}$$

$$b(\mathbf{u}, \mu) = 0 \quad \forall \mu \in \mathbf{M}, \tag{68}$$

*where the bivariate form* $a(.,.)$ *is assumed to be linear with respect to its second argument only, and* $b(.,.)$ *is a bilinear form. Assume that the following conditions are satisfied*
**(A1)** $\mathbf{f} \in \mathbf{X}^\star$,
**(A2)** $\exists \beta_2 = \text{const} > 0$ *with*

$$|b(\mathbf{v}, \mu)| \leq \beta_2 ||\mathbf{v}||_{\mathbf{X}} ||\mu||_{\mathbf{M}} \quad \forall \mathbf{v} \in \mathbf{X}, \forall \mu \in \mathbf{M},$$

**(A3)** *the LBB (Ladyženskaya-Babuška-Brezzi) condition, i.e.*
$\exists \beta_1 = \text{const} > 0$ *such that*

$$\sup_{\mathbf{v} \in \mathbf{X}, \mathbf{v} \neq \mathbf{0}} \frac{b(\mathbf{v}, \mu)}{||\mathbf{v}||_{\mathbf{X}}} \geq \beta_1 ||\mu||_{\mathbf{M}} \quad \forall \mu \in \mathbf{M}, \tag{69}$$

**(A4)** *the strong* $\mathbf{V}_0$ *– monotonicity of* $a$, *i.e.* $\exists \alpha_1 = \text{const} > 0$ *with*

$$a(\mathbf{u}, \mathbf{u} - \mathbf{v}) - a(\mathbf{v}, \mathbf{u} - \mathbf{v}) \geq \alpha_1 ||\mathbf{u} - \mathbf{v}||^2_{\mathbf{X}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}_0 \tag{70}$$

*where*

$$\mathbf{V}_0 = \mathbf{V}(0) = \{\mathbf{v} \in \mathbf{X} : b(\mathbf{v}, \mu) = 0 \quad \forall \mu \in \mathbf{M}\}, \tag{71}$$

**(A5)** *and the Lipschitz continuity of* $a$, *i.e.*, $\exists \alpha_2 = \text{const} > 0$ *with*

$$|a(\mathbf{u}, \mathbf{w}) - a(\mathbf{v}, \mathbf{w})| \leq \alpha_2 ||\mathbf{u} - \mathbf{v}||_{\mathbf{X}} \, ||\mathbf{w}||_{\mathbf{X}} \quad \forall \mathbf{w} \in \mathbf{X} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}.$$

*Then there exists a unique solution* $(\mathbf{u}, \lambda) \in \mathbf{X} \times \mathbf{M}$.

**Proof.** ¿From **(A4)** and **(A5)** it follows that the nonlinear problem
*Find* $\mathbf{u} \in \mathbf{V}_0$ *with*

$$a(\mathbf{u},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{V}_0 \tag{72}$$

has a unique solution $\mathbf{u} \in \mathbf{V}_0$ (see, e.g., [19, Theorem 24.2.]). Then we define $\mathbf{l} \in \mathbf{X}^\star$ by

$$\langle \mathbf{l},\mathbf{v}\rangle = \langle \mathbf{f},\mathbf{v}\rangle - a(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}. \tag{73}$$

Since

$$\langle \mathbf{l},\mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in \mathbf{V}_0,$$

we get $\mathbf{l} \in {\mathbf{V}_0}^0$, and with Lemma 7 (3°) it follows that there exists a unique $\lambda \in \mathbf{M}$ with

$$b(\mathbf{v},\lambda) = \langle \mathbf{l},\mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{74}$$

Finally, we conclude from $\mathbf{u} \in \mathbf{V}_0$, (73), (74), that the pair $(\mathbf{u},\lambda) \in \mathbf{X}\times\mathbf{M}$ fulfills (67), (68). ∎

Now, we apply Theorem 5 to the variational formulation **(VF2)**.

**THEOREM 6** *Suppose that* $\mathbf{f} \in \mathbf{Y}^\star$ *and that the material function fulfills (50) and* **(M1)**, **(M2)**, **(M4)** *and* **(M5)**. *Then (55), (56) have a unique solution.*

**Proof.** The assumptions **(A1)**, **(A2)** and **(A3)** are verified in the proof of Theorem 2. Further, **(A4)** and **(A5)** follow from the Lemmata 5 and 6. ∎

The results of Section 3.4 can be carried over to the nonlinear case without modification.

**THEOREM 7** *Assume that* $\Omega$ *is bounded, Lipschitz-continuous and simply-connected. Assume further that* $\mathbf{f} \in H(\operatorname{div};\Omega)$ *is divergence-free, i.e.,*

$$\operatorname{div}\mathbf{f} = 0. \tag{75}$$

*Then, the unique solution* $(\mathbf{u},\lambda)$ *of*

$$\begin{aligned} \int_\Omega \nu(|\,\mathbf{rot}\,\mathbf{u}|)\,\mathbf{rot}\,\mathbf{u}\cdot\mathbf{rot}\,\mathbf{v}\,dx + \int_\Omega \rho\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,dx - \int_\Omega \lambda\operatorname{div}\mathbf{v}\,dx &= \int_\Omega \mathbf{f}\,\mathbf{v}\,dx \quad \forall \mathbf{v}\in\mathbf{Y} \\ -\int_\Omega \mu\operatorname{div}\mathbf{u}\,dx &= 0 \quad \forall \mu \in \mathbf{M} \end{aligned}$$

*fulfills*

$$\int_\Omega \nu(|\,\mathbf{rot}\,\mathbf{u}|)\,\mathbf{rot}\,\mathbf{u}\cdot\mathbf{rot}\,\mathbf{v}\,dx = \int_\Omega \mathbf{f}\,\mathbf{v}\,dx \quad \forall \mathbf{v}\in\mathbf{Y}. \tag{76}$$

**Proof.** Coincides with that of Theorem 4. Indeed, the only term in (35) involving $\nu$ is not changed during the proof of Theorem 4. ∎
Thus, the variational formulation of the nonlinear problem is adequate to the physical problem. Again, we have $\lambda = 0$ such that we can omit the $\lambda$-term and solve the coercive problem:
**(VF2$^\star$)** *Find* $\mathbf{u} \in \mathbf{Y}$ *such that*

$$\int_\Omega \nu(|\,\mathbf{rot}\,\mathbf{u}|)\,\mathbf{rot}\,\mathbf{u}\cdot\mathbf{rot}\,\mathbf{v}\,dx + \int_\Omega \rho\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,dx = \int_\Omega \mathbf{f}\,\mathbf{v}\,dx \quad \forall \mathbf{v}\in\mathbf{Y}. \tag{77}$$

The unique solution $\mathbf{u}$ of (VF2$^\star$) coincides with the $\mathbf{u}$ in the solution of (VF2).

# 5 Conclusions

We have established mixed variational formulations for linear and nonlinear three-dimensional static electro-magnetic field problems. We have shown that the formulation in the space $\mathbf{Y} = H_0(\,\mathbf{rot}\,;\Omega) \cap H(\,\mathrm{div}\,;\Omega)$ represents the magnetostatic problem well. The analysis shows that the both the linear and the nonlinear problem have a unique solution. Further, the solution for the Lagrange multiplier $\lambda$ is equal to zero. Thus, the solution for the variable $\mathbf{u}$ (in which we are interested only) can be obtained by solving a coercive formulation not involving the unknown $\lambda$.

# References


[1] D. N. Arnold, R. S. Falk, and R. Winther. Preconditioning in $H(\mathrm{div})$ and applications. *Mathematics of Computation*, 66, 1997. To appear. Preprint available via http://www.math.psu.edu/dna/papers/hdiv.ps or hdiv.dvi.

[2] O. Bíró. *Numerische Aspekte von Potentialformulierungen in der Elektrodynamik.* Habilitationsschrift, Technische Universität Graz, 1992.

[3] D. Braess. *Finite Elemente - Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie.* Springer–Lehrbuch. Springer Verlag, Berlin, Heidelberg, 1992.

[4] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods.* Number 15 in Springer Series in Computational Mathematics. Springer–Verlag, Berlin, 1991.

[5] M. Costabel and M. Dauge. Singularities of Maxwell's equations on polyhedral domains. Prépublication 96-30, Institut de Recherche Mathématique de Rennes, December 1996.

[6] V. Girault and P.-A. Raviart. *Finite Element Approximation of the Navier-Stokes Equations.* Number 749 in Lecture Notes in Mathematics. Springer–Verlag, Berlin, 1979.

[7] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations.* Number 5 in Springer Series in Computational Mathematics. Springer–Verlag, Berlin, 1986.

[8] B. Heise. Analysis of a fully discrete finite element method for a nonlinear magnetic field problem. *SIAM J. Numer. Anal.*, 31(3):745–759, 1994.

[9] B. Heise. A mixed variational formulation for 3D magnetostatics and its finite element discretization. Technical Report 96-6, Universität Linz, Institut für Mathematik, Arbeitsgruppe Numerische Mathematik und Optimierung, 1996.

[10] B. Heise and M. Kuhn. Parallel solvers for linear and nonlinear exterior magnetic field problems based upon coupled FE/BE formulations. *Computing*, 56(3):237–258, 1996.

[11] A. Kameari and K. Koganezawa. Convergence of ICCG method in FEM using edge elements without gauge condition. 1996. Proc. of the IEEE CEFC Conference, Okayama, Japan, March 18 - 20, 1996.

[12] R. C. Mesquita. Additional properties of the incomplete gauge formulation for 3-D finite-element magnetostatics. *IEEE Transactions on Magnetics*, MAG-30:2861–2864, 1994.

[13] I. Nishiguchi, H. Matsuzawa, and M. Sasaki. A comparative study of magnetostatics using three-field formulation and conventional vector potential formulations. 1996. Proc. of the IEEE CEFC Conference, Okayama, Japan, March 18 - 20, 1996.

[14] K. Preis, I. Bardi, O. Biro, C. Magele, G. Vrisk, and K. R. Richter. Different finite element formulations of 3D magnetostatic fields. *IEEE Transactions on Magnetics*, MAG-28:1056–1059, 1992.

[15] D. Rodger. Finite-element method for calculating power frequency 3-dimensional electromagnetic field distributions. *IEE Proceedings - A*, 130:233–238, 1983.

[16] D. Rodger and P. J. Leonard. Modelling the electromagnetic performance of moving rail gun launchers using finite elements. *IEEE Transactions on Magnetics*, 29:496–498, 1993.

[17] Y. G. Soloveitchik. Mathematical model of three-dimensional electro-magnetic fields in electrotechnical devices. Preprint UDK 621.37, Novosibirsk Electrotechnical Institute, 1990. In Russian.

[18] S. Williamson and E. K. C. Chan. Three-dimensional finite-element formulation for problems involving time-varying fields, relative motion, and magnetic saturation. *IEE Proceedings - A*, 140:121–130, 1993.

[19] E. Zeidler. *Vorlesungen über nichtlineare Funktionalanalysis II: Monotone Operatoren*, volume 9 of *Teubner–Texte zur Mathematik*. Teubner–Verlag, Leipzig, 1977.